CONNECTED SUM OF COMPACT ANALYTIC HYPERSURFACES

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Abstract. A construction is given to find a global analytic [resp. algebraic] equation for the connected sum of two compact, analytic [resp. algebraic] hypersurfaces having global equation. As an application we give explicit algebraic equations for a model of the connected sum of k copies of $\mathbb{RP}^3$ in $\mathbb{R}^5$.

Introduction

Methods to obtain algebraic models of the connected sum of manifolds are well known (see [3]), but the dimension of the euclidean space where such models exist is usually very high. In the first section of this paper we give a construction that, in the case of compact hypersurfaces, produces a model in the same euclidean space. Moreover, the degree of the model is exactly the sum of the degrees of the two given hypersurfaces.

Since the proofs involve only the fact that the given hypersurfaces are analytic, the statements are all given in the C^ω case.

In the second section we use such a construction to find explicit algebraic models of the connected sum of k copies of $\mathbb{RP}^3$'s in $\mathbb{R}^5$. Clearly such models do exist, in fact any compact, orientable 3-manifold can be C^∞-embedded in $\mathbb{R}^5$ [6] hence, by Theorem 14.1.4 of [2], it can be isotopically approximated by a smooth algebraic variety in $\mathbb{R}^5$. This means, in particular, that any compact, orientable 3-manifold has an algebraic model in $\mathbb{R}^5$. In the case of the connected sum $\#_k\mathbb{RP}^3$ it is easily seen by an argument due to Whitney [10] that 5 is the smallest embedding dimension.

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1. Connected sum of hypersurfaces

Let M be be an analytic set and let $N \subset M - \partial M$ be a hypersurface We will say that N is a *regular hypersurface* if there exists a C^ω function $f : M \to \mathbb{R}$ such that 0 is a regular value for f and $N = f^{-1}(0)$ (in particular $N \subset M - \operatorname{Sing} M$). Such an f will be called an *equation* for N. Clearly the expression "regular hypersurface" is adopted only for convenience and to remind that one is speaking about the inverse image of a regular value of a function.

Let N_1 and $N_2 \subset M$ be regular hypersurfaces in M and let f_1, f_2 be proper equations for N_1 and N_2 respectively. Suppose that $N_1 \cap N_2$ is constituted by a single point p.

Lemma 1.1. *Let N_2' be the connected component of N_2 containing p and call $A_+ = \{f_1 > 0\}$ and $A_- = \{f_1 < 0\}$. In the above situation N_1 and N_2 are tangent at p and $N_2' - \{p\}$ is contained either in A_+ or A_-.*

Proof: $N_2' - \{p\}$ is connected (in the case $\dim N_2 = 1$ use the fact that N_2 is compact hence $N_2' \cong S^1$) so it must be contained in one of A_+ or A_-. From this follows that N_1 and N_2 are tangent at p. In fact if it were not then $N_2(p) - \{p\}$ should have points in both A_+ and A_-. $\square$

So, without loss in generality, we can suppose $df_1(p) = -df_2(p)$. For all $\varepsilon \in \mathbb{R}$ let N_ε denote the set defined by the equation $g_\varepsilon = f_1 f_2 - \varepsilon = 0$. We will prove the following:

Proposition 1.2. *There exists $\delta > 0$ such that for all $0 < \varepsilon < \delta$ N_ε is a regular hypersurface diffeomorphic to a connected sum $N_1 \# N_2$ and g_ε is a proper, regular equation for it.*

Remark. Suppose that N_1 and N_2 are smooth complete intersections of codimension k, and suppose that they are defined respectively by $f_1, g_2, \ldots, g_k$ and $f_2, g_2, \ldots, g_k$ (i.e. $k-1$ equations are the same for both the N_i), being f_i proper. Suppose finally that they intersect at a single point p and that $df_i(p)$ have opposite directions. Then for $\varepsilon > 0$ small enough the set defined by $f_1 f_2 - \varepsilon$, $g_2, \ldots, g_k$ is diffeomorphic to a connected sum $N_1 \# N_2$.

To see this, it is enough to apply Proposition 1.2 considering the analytic set M' defined by $g_2, \ldots, g_k$ as the ambient space (see Fig. 1).

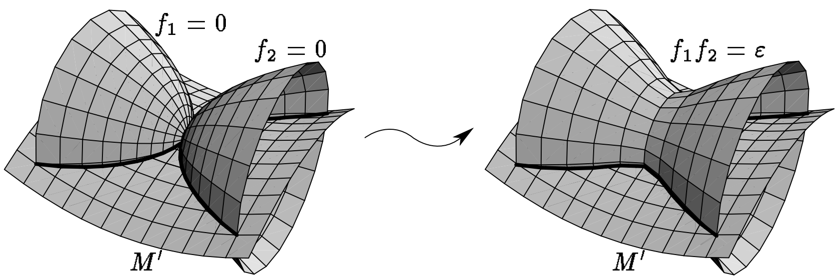

Fig. 1

Proof of Proposition 1.2. First we prove that for $\varepsilon \neq 0$ small enough, N_ε is regular and $f_1 f_2 - \varepsilon$ is a regular equation for it. To see this, note that $f_1 f_2$ has no critical points on $N_0 = N_1 \cup N_2$ other then p, hence for all neighborhood U of p there exists a neighborhood V of $N_0 - U$, that contains no critical points for $f_1 f_2$. Since f_1 and f_2 are proper functions, their product $f_1 f_2$ is proper too, hence $N_\varepsilon \subset U \cup V$ for ε small enough. So it is enough to show that we can choose U containing no critical points other then p, that is p is an isolated critical point for $f_1 f_2$. But this follows by an easy application of the curve selection lemma (see [7], [1] for the lemma in the algebraic case or [8] for the general case).

Since $df_1(p) = -df_2(p) \neq 0$ and since, by Lemma 1.1, N_1 and N_2 stand on opposite sides at p, we can find a neighborhood U of p and analytic coordinates $x = (x_1, \dots, x_n)$, y on U, centered at p and such that $f_1(x,y) = h(x) + y$ and $f_2(x,y) = q(x,y)\big(h(x) - y\big)$, where h is a non negative analytic function vanishing only at 0, and q is a positive analytic function. Hence in such coordinates $f_1 f_2(x,y) = q(x,y)\big(h^2(x) - y^2\big)$.

Let c be a positive number and $U' = \big\{(x,y) \bigm| |y| \leq c\big\}$. U' is an analytic manifold with boundary and the two functions $g_1, g_2 : U' \to \mathbb{R}$ defined by $g_1(x,y) = h^2(x) - y^2$ and $g_2(x,y) = f_1 f_2(x,y) = q(x,y)\big(h^2(x) - y^2\big)$ have the same sign and the same zero set. Is a well known fact (see for instance [1] for the algebraic case and [8] for the general case) that in such hypothesis there exists $\delta > 0$ such that for all $0 < \varepsilon \leq \delta$ the sets

$$V_1 = \big\{(x,y) \bigm| g_1(x,y) \leq \varepsilon\big\}, \quad V_2 = \big\{(x,y) \bigm| g_2(x,y) \leq \varepsilon\big\}$$

are diffeomorphic.

The same argument shows that, for ε and c small enough, the set $\big\{x \bigm| h^2(x) \leq c^2 + \varepsilon\big\}$ is diffeomorphic to the disk D^n. Since the functions y and its restriction to the boundary of V_2 have no critical points on V_2, then V_2 is a cylinder, that is to say, V_1 and V_2 are both diffeomorphic to $D^n \times [-c, c]$.

Finally, observe that $\partial f_1 f_2 / \partial x$ is not zero at all points of N_0 other then p, hence, for ε small enough, the manifold N_ε is transversal to the the boundary of U'. This implies that $N_\varepsilon - \mathring{U}'$ is diffeomorphic to $N_0 - \mathring{U}'$. Moreover, for c small enough, the two sets $N_i \cap U'$ are both diffeomorphic to the set $\big\{x \bigm| h(x) \leq c^2\ \big\}$, which is diffeomorphic to the disk D^n.

This is enough to say that N_ε is diffeomorphic to the manifold obtained gluing $N_1 - \mathring{U}'$ and $N_2 - \mathring{U}'$ along their boundaries with a diffeomorphism, which extends to a diffeomorphism of the two disks $N_i \cap U'$, that is to say N_ε is diffeomorphic to a connected sum $N_1 \# N_2$. □

Remark. In the case that N_1 and N_2 are both orientable, observe that if they are oriented in such a way that the gluing diffeomorphism is orientation reversing, then N_ε is just the oriented connected sum. So if $M - \operatorname{Sing} M$ is oriented, and N_1 and N_2 are oriented according to their equations (i.e. the gradient of the equation, with respect to some Riemannian metric on $M - \operatorname{Sing} M$, and an oriented bases of the tangent space to the hypersurface are an oriented bases of the tangent space to $M - \operatorname{Sing} M$ at each point of the hypersurface) then the resulting hypersurface N_ε, equipped with the orientation induced by the equation g_ε, is diffeomorphic to the oriented connected sum $N_1 \# N_2$.

Remark. In the case of hypersurfaces N_1, N_2 of Euclidean space fulfilling the hypothesis of Proposition 1.2, the construction gives a model of $N_1 \# N_2$ embedded in the same Euclidean space. Furthermore, in the case of algebraic hypersurfaces of degree d_1 and d_2 the construction gives a model for $N_1 \# N_2$ of degree $d_1 + d_2$.

Remark. Finally observe that whenever two compact hypersurfaces N_1, $N_2 \subset \mathbb{R}^{n+1}$ are given, one can find new ones N_1', N_2' such that their intersection is a single point and N_i' is diffeomorphic to N_i. This fact can be seen as follows. Chose a unit vector v such that the function $h_v : x \mapsto \langle v, x \rangle$ has an isolated maximum point p_1 on N_1 (in fact, since N_1 is compact, there exists an open and dense subset $U \subseteq S^n$ such that h_v is a Morse function for N_1, see for instance [5]). Let p_2 a minimum point for $h_v|_{N_2}$ and let $d_1 = \min\{||x_1 - p|| \mid x \in N_1,\ h_v(x) = h_v(p_1),\ x \neq p_1\}$ and $d_2 = \max\{||x - p_2|| \mid x \in N_2\}$. If τ is the translation $x \mapsto x + p_2 - p_1$ and λ is the dilatation $x \mapsto p_2 + (x - p_2)d_1/2d_2$, then it is easy to check that $N_1' = \tau(N_1)$ and $N_2' = \lambda(N_2)$ are in the desired position ($N_1' \cap N_2' = \{p_2\}$). Observe that, in the algebraic case, this procedure does not change the degree of the equations.

2. Connected sum of $\mathbb{RP}^3$'s

As announced in the introduction, we apply the construction of §1 to find an explicit algebraic model of the connected sum of $\mathbb{RP}^3$'s in $\mathbb{R}^5$. As already observed, such a model does exist and $\mathbb{R}^5$ is in fact the smallest euclidean space where it can be found. This follows by a classical result due to Whitney [10] saying that all sections of the normal bundle to an embedding of $\mathbb{RP}^2$ in $\mathbb{R}^4$ has a zero, hence:

Proposition 2.1. *Every 3-manifolds containing an embedded projective plane cannot be embedded into $\mathbb{R}^4$.*

Proof: If such an embedding existed, its normal bundle should be trivial, giving a never zero section of the normal bundle of the $\mathbb{RP}^2$ contained in the 3-manifold. □

In particular the connected sum of $\mathbb{RP}^3$'s cannot be embedded in $\mathbb{R}^4$.

To construct an algebraic model of $\mathbb{RP}^3 \# \mathbb{RP}^3$ in $\mathbb{R}^5$, we start from a well known model of $\mathbb{RP}^3 \subset \mathbb{R}^5$. $\mathbb{RP}^3$ is diffeomorphic to the unitary tangent bundle of the sphere S^2 (see for instance [9]), hence it has obvious equations in $\mathbb{R}^6 = \mathbb{R}^3_x \times \mathbb{R}^3_y$:

$$||x||^2 = 1, \qquad ||y||^2 = 1, \qquad \langle x, y \rangle = 0. \tag{2-1}$$

This means that $\mathbb{RP}^3$ is an algebraic subset of $S^2 \times S^2$, but $S^2 \times S^2$ is algebraically embedded in $\mathbb{R}^5$ (as the hypersurface obtained by rotating a sphere S^2 around a 2-plane), hence the same holds for $\mathbb{RP}^3$. By an easy calculation one can write down the following equations for $\mathbb{RP}^3 \subset \mathbb{R}^5$:

$$(||x||^2 - (R^2 + r^2))^2 + 4R^2(x_4^2 + x_5^2) = 4R^2r^2, \tag{2-2}$$

$$x_1(||x||^2 - (R^2 + r^2)) + 2Rx_2x_4 + 2Rx_3x_5 = 0. \tag{2-3}$$

The former is the equation of the torus $S^2 \times S^2$, spanned by the rotation of the 2-sphere of radius r, centered at $(R, 0, 0, 0, 0)$ and contained in the 3-space of coordinates (x_1, x_4, x_5) around the 2-space of coordinates (x_4, x_5), the latter is derived from the third of (2-1). More precisely, consider the mapping $\Phi : \mathbb{R}^6 \to \mathbb{R}^5$

$$\Phi(x, y) = (x_1(R + ry_1), x_2(R + ry_1), x_3(R + ry_1), ry_2, ry_3)$$

then $\Phi|_{S^2 \times S^2}$ is a diffeomorphism with the hypersurface defined by (2-2), and the second equation is the push forward via Φ of the third of the (2-1).

Call $T \subset \mathbb{R}^5$ the algebraic variety defined by (2-2), M the one defined by (2-3) and $N = T \cap M$. Is not hard to see that the gradients of the two equations are independent at each point of N, that is N is a smooth subvariety of $\mathbb{R}^5$ which is a complete intersection.

Remark. It is also easy to prove that the mapping Φ is a biregular isomorphism between $\mathbb{RP}^3 \subset \mathbb{R}^6$ and N, and since the embedding of $\mathbb{RP}^3$ in $\mathbb{R}^6$ is biregular, then N is a biregular embedding of $\mathbb{RP}^3$ in $\mathbb{R}^5$.

The idea to construct equations for $\mathbb{RP}^3 \# \mathbb{RP}^3$ is quite simple. Let $p \in N$, suppose to be able to construct another torus T_1 such that the following hold:

(A) $T \cap T_1 = \{p\}$, that is the tori are tangent at p;
(B) $N_1 = T_1 \cap M$ is diffeomorphic to $\mathbb{RP}^3$.

Let f be the function defining T and f_1 the one defining T_1, then, according to Proposition 1.2 and the subsequent remark, for ε small enough the equation

$$ff_1 - \varepsilon = 0 \tag{2-4}$$

will define in M a manifold diffeomorphic to $N \# N_1$, that is (2-4) and (2-3) are algebraic equations for $\mathbb{RP}^3 \# \mathbb{RP}^3 \subset \mathbb{R}^5$.

The problem is so reduced to find a torus T_1 fulfilling conditions (A) and (B). In order to do this we prove that we can approximate the torus T by a family of tori T_ε that are all tangent to T at a fixed point $p \in T$; by transversality arguments, it will follow that, for ε small enough, the tori belonging to such a family definitively satisfy the desired conditions.

Proposition 2.2. *Let $p \in T$, then there exists a family $\{T_\varepsilon\}_{\varepsilon > 0}$ of tori such that:*

(1) *$T_\varepsilon \cap T = \{p\}$ (i.e. T_ε is tangent to T at p);*
(2) *for all neighborhood U of T there exist $\delta > 0$ such that $T_\varepsilon \subset U$ for all $0 < \varepsilon < \delta$.*

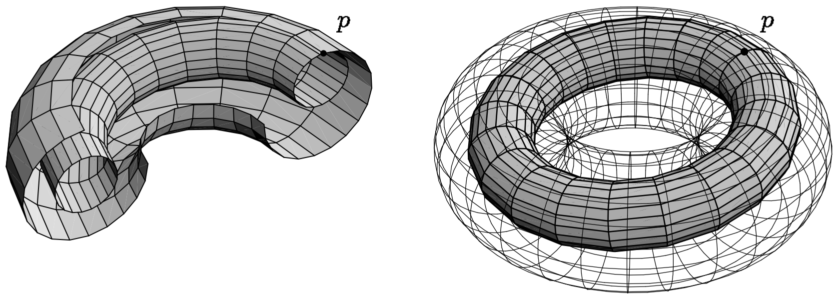

FIG. 2

Proof: Let $c^+ = (R, 0, 0, 0, 0)$, $c^- = (-R, 0, 0, 0, 0)$ and let S^+, S^- be the 2-spheres of radius r, centered at c^+, c^- respectively and contained in the 3-space E, of coordinates (x_1, x_4, x_5). It is clear that $S^+, S^- \subset T$, and T is spanned by the rotation of either S^+ or S^- around the 2-space A of coordinates (x_4, x_5) (i.e. T is the orbit of either S^+ or S^- under the action of the group of rotations of $\mathbb{R}^5$ leaving A fixed). Without loss in generality we can suppose $p \in S^+$. For all $\varepsilon \in (0, r)$ let S^+_ε be the 2-sphere in E of radius $r - \varepsilon$ and centered at $c^+ + \varepsilon(p - c^+)/r$ (i.e. S^+_ε is inside S^+ and tangent at S^+ in p) and S^-_ε the 2-sphere of radius $r - \varepsilon$ centered at c^-. Finally let $B_\varepsilon \subset E$ be the 2-plane of symmetry of $S^+_\varepsilon, S^-_\varepsilon$ and let T_ε be the torus spanned by the rotation of either S^+_ε or S^-_ε around B_ε. We claim that the family $\{T_\varepsilon\}$ has the required properties (see Fig. 2).

Clearly, since $S^+_\varepsilon \to S^+$ and $B_\varepsilon \to A$ as $\varepsilon \to 0$, (2) follows by compactness arguments.

For what concern (1), we remark that T_ε is a union of S^2 spanned by the rotation of either a point $p^+ \in S^+_\varepsilon$ or $p^- \in S^-_\varepsilon$ around their axis B_ε, hence the thesis is a consequence of the following lemma:

Lemma 2.3. *Let $p^+ \in E$ be a point inside S^+, $p^- \in E$ a point inside S^- and let $B \subset E$ be the axis of the segment joining p^+ and p^-. Let $S \subset \mathbb{R}^5$ be the 2-sphere spanned by the rotation around B of either p^+ or p^-. Then S is contained inside T. Moreover, if $p^+ \in S^+$ then S is tangent to T at p^+.*

Proof: Suppose that p^+ is inside S^+. Let $X \in S$, then $X = \sigma(p^+)$, being σ a rotation of $\mathbb{R}^5$ leaving B fixed. Let τ be a rotation such that $\tau(X) \in E \cap \{x_1 \geq 0\}$ and leaving A fixed (clearly such a τ does exist). Let us call $Y = \tau(X)$. It is obvious that to prove the lemma is equivalent to show that Y is inside S^+.

Let $x_i(X)$ denote the i^{th} coordinate of X, then $(x_1(X), x_4(X), x_5(X)) \in E$ belongs to the segment joining p^+ to p^- and, since the rotation τ lives A fixed, $(x_4(Y), x_5(Y)) = (x_4(X), x_5(X))$ hence Y belongs to the cylinder C parallel to the x_1 axis and based at the segment joining $(0, x_4(p^+), x_5(p^+))$ to $(0, x_4(p^-), x_5(p^-))$.

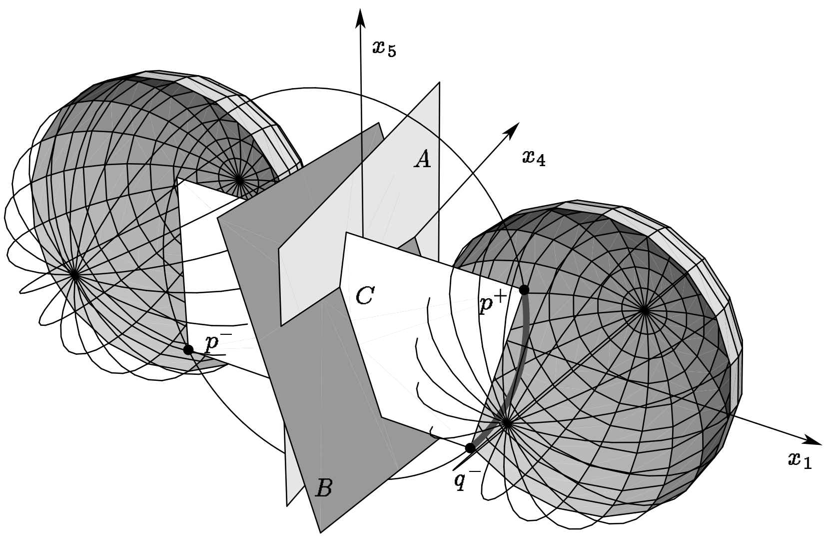

FIG. 3

Let q^- denote the symmetric of p^- in E with respect to A. Suppose that $A \cap B$ is a line, since $Y = \tau\sigma(p)$ and $\tau\sigma$ leaves $A \cap B$ fixed then Y belongs to the circular cylinder in E having $A \cap B$ as axis and passing through p^+. Together with what we proved above, this means that Y belongs to the arc of the circle passing through p^+, p^- and q^-, and joining p^+ to q^-, from which the thesis follows.

If A and B are parallel or equal, it is easily seen that the point Y belongs to the segment joining p^+ to q^-. Once again the thesis follows.

Finally, if $p^+ \in S^+$, observe that $X \in T$ if and only if $Y \in S^+$. But the arc containing Y touches S^+ exactly at p^+, and $Y = \tau\sigma(p^+) = p^+$ implies that σ is the identity, hence $X = p^+$. $\square$

This ends the proof of Proposition 2.2. $\square$

Remark. Lemma 2.3 can be obviously restated for the embedding $S^p \times S^q \subset \mathbb{R}^{p+q+1}$ obtained by the rotation of an S^q around a q-plane, and the same kind of arguments, used in the above proof, proves this general case too.

Clearly the torus T_ε constructed in Proposition 2.2 is obtained from T changing the parameters R and r and acting on it by an isometry of $\mathbb{R}^5$, hence, if $f(R, r, x)$ denotes the function defining the torus T, that is the left hand side of equation (2-2), then T_ε has an equation of the form $f(R_\varepsilon, r_\varepsilon, \varphi_\varepsilon(x)) = 0$, where

$$R_\varepsilon = \frac{1}{2}\left\| c^+ - c^- + \frac{\varepsilon}{2}(p - c^+) \right\|, \quad r_\varepsilon = r - \varepsilon,$$

and φ_ε is an isometry of $\mathbb{R}^5$, which is a C^∞ function of ε. We now prove the following proposition:

Proposition 2.4. *If ε is sufficiently small, then $T_\varepsilon \cap M$ is diffeomorphic to $\mathbb{RP}^3$.*

Proof: Let $\widetilde{M}$ denote $M - U$, being U a small ball around 0 such that T is far from $\partial(M - U)$. Consider the mapping $F : \widetilde{M} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ defined by:

$$F(x, \xi) = \big(f(R_\xi, r_\xi, \varphi_\xi(x)), \xi\big).$$

F is of class C^∞, for ξ sufficiently small, and proper. Furthermore $F^{-1}(0, \varepsilon) = (T_\varepsilon \cap M) \times \{\varepsilon\}$. Since $T_0 = T$ and $T \cap M$ is a complete intersection, $(0,0)$ is a regular value for F. By Ehresmann's fibration theorem (see for instance [4] or [5]), if ε is sufficiently small, $F^{-1}(0, \varepsilon)$ is diffeomorphic to $F^{-1}(0,0) = (T_0 \cap M) \times \{0\}$. Since $T_0 = T$ is diffeomorphic to $\mathbb{RP}^3$, the thesis follows. $\square$

Proposition 2.2 and Proposition 2.4 ensures that one can find a torus T_1 such that (A) and (B) hold, hence one can find algebraic equations for $\mathbb{RP}^3 \# \mathbb{RP}^3$ in $\mathbb{R}^5$.

Remark. Let f and f_1 be the equations of T and T_1 respectively. By the way we constructed T_1, the equation f_1 has the same degree as f, that is 4. Hence the two equations we found, that is (2-3) and (2-4) have degree respectively 3 and 8, in other words we found $\mathbb{RP}^3 \# \mathbb{RP}^3$ as a smooth algebraic subset of $\mathbb{R}^5$ of degree 24.

It is clear from the proof in §1, Proposition 1.2 can be generalized as follows:

Proposition 2.5. *Let $N_1, \dots, N_k \subset M$ be regular hypersurfaces, and $f_1, \dots, f_k$ proper, regular equations for them. Suppose that*

$$N_i \cap N_j = \begin{cases} \{p_i\} & \text{if } |i-j| = 1 \\ \varnothing & \text{otherwise} \end{cases} \quad \text{and} \quad df_i(p_i) = -df_{i+1}(p_i) \quad \forall\, i = 1, \dots, k-1.$$

Then there exists $\delta > 0$ such that for all $0 < \varepsilon < \delta$ the set $N_\varepsilon = \{x \in M \mid g_\varepsilon(x) = 0\}$, where $g_\varepsilon = f_1 \cdots f_k - \varepsilon$, is a regular hypersurface diffeomorphic to the connected sum $N_1 \# \cdots \# N_k$ and g_ε is a proper, regular equation for it.

Remark. Using this proposition one can iterate the construction of the torus T_1, to find algebraic equations for the connected sum $\#_k \mathbb{RP}^3$ in $\mathbb{R}^5$. More precisely let $p_1 \in N$ and T_1 be as above, that is $T_1 \cap M$ is diffeomorphic to $\mathbb{RP}^3$, and T_1 is tangent to T at p_1. Now, let $p_2 \in T_1 \cap M$, $p_2 \neq p_1$ the same arguments used to construct T_1 can be used to construct a torus T_2 such that $T_2 \cap M$ is diffeomorphic to $\mathbb{RP}^3$ and T_2 is tangent to T_1 at p_2. This process can be iterated k times. Observe that each time a new torus is constructed, it is inside the previous one (except for the tangency point), hence $T_i \cap T_j = \varnothing$ if $|i-j| \geq 2$. By Proposition 2.5, if $f, f_1, \dots, f_{k-1}$ are the equations of these tori, than for a suitable choice of ε, $ff_1 \cdots f_{k-1} - \varepsilon$ together with (2-3) are equations for the connected sum of k copies of $\mathbb{RP}^3$'s. Once again all the equation f_i have the same degree as f, hence we find $\#_k \mathbb{RP}^3$ in $\mathbb{R}^5$ as a smooth algebraic surface of degree $12k$, which is complete intersection.

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