SURFACES WITH ASSIGNED APPARENT CONTOUR

DOMENICO LUMINATI

Introduction

In all the paper we shall denote by $\mathbf{S}^1$ the unitary circle in Euclidean plane $\mathbb{R}^2$ and by $\coprod_{h=1}^k \mathbf{S}^1$ the disjoint union of k identical copies of the circle; all surfaces and maps will be supposed to be smooth (i.e. of class C^∞). Furthermore we shall often use "cut and paste" techniques, that work nicely in C^0 or PL category. Since in low dimension these categories are the same as the C^∞ one, we shall not care to give technical details.

Let S and $\mathbf{N}$ be smooth, compact surfaces, $p \in S$.

Definition. A map $F : S \to \mathbf{N}$ is said to be excellent at p if its germ at p is equivalent (left-right) to one of the following three normal forms:

i) $(x, y) \mapsto (x, y)$, *ii)* $(x, y) \mapsto (x, y^2)$, *iii)* $(x, y) \mapsto (x, y^3 - xy)$.

Clearly, if F is excellent at all p, then the set Σ_F of its critical points is a smooth curve in S (i.e. Σ_F is a finite union of disjoint circles).

Definition. F is said to be excellent if the following two hold:

(1) F is excellent at all $p \in S$;
(2) the apparent contour of F (i.e. the set $\Gamma_F = F(\Sigma_F) \subset \mathbf{N}$) is a smooth curve except for a finite number of singularities of the following two local kinds:

Crossing: $\big\{(x, y) \in \mathbb{R}^2 \bigm| xy = 0\big\}$; *Cusp:* $\big\{(x, y) \in \mathbb{R}^2 \bigm| x^3 - y^2 = 0\big\}$.

A classical theorem of Whitney [22] asserts that excellent maps are stable and constitute an open, dense subset of the set of C^∞ maps between two surfaces. This is the reason why excellent maps are often called generic.

Given a curve $\Gamma \subset \mathbf{N}$ satisfying condition (2) of the previous definition, we ask whether a surface S and an excellent map $F : S \to \mathbf{N}$ exist such that $\Gamma_F = \Gamma$. Moreover we should like to know how many, up to right equivalence, such maps exist (recall that $F_1, F_2 : S \to \mathbf{N}$ are said to be right equivalent if there exists a diffeomorphism $\varphi : S \to S$ such that $F_2 = F_1 \circ \varphi$). This problem can be restated as follows: given a curve, say $f : \coprod_{h=1}^k \mathbf{S}^1 \to \mathbf{N}$, with only cusp and normal crossings (Definition 1.1) find out all surfaces $S \supset \coprod_{h=1}^k \mathbf{S}^1$ and a maps $F : S \to \mathbf{N}$ such that $\Sigma_F = \coprod_{h=1}^k \mathbf{S}^1$ and $F\big|_{\coprod_{h=1}^k \mathbf{S}^1} = f$.

The basic idea, that will be used to solve this problem, is not dissimilar from the one used by Francis and Troyer [8], [9] to solve the problem for plane curves and arises from a very simple remark by Haefliger [11]: let F be an excellent extension of f and U a tubular neighborhood of Σ_F, then F restricted to $S-U$ is an immersion. Furthermore, one can suppose that F restricted to ∂U is a curve with only normal crossings. Hence the problem reduces to the following two sub–problems: *i)* find out local, excellent extensions of f to a union of cylinders and Möbius bands, *ii)* find out immersions that extend the curves resulting as "boundary of the local extensions", and the last problem can be solved using Blank's methods [1], [19].

In §1 we define an extension of Blank's word for curves with cusps and normal crossings, from which, by a purely combinatorial algorithm, we construct a set (the set of minimal assemblages), which is in one to one correspondence with the set of excellent extensions of the curve, up to right equivalence (Theorem 1.34).

In §0 we sketch, without any proof, the methods introduced by Blank [1], [19], and subsequently developed by other authors ([14], [4], [5], [10], [3], [2]), to find all (up to right equivalence) the extensions, that are immersions, of a curve with only normal crossings. We include this section because our notations and statements differs slightly from those in the original papers. For a complete survey on this matter see also the first chapter of [13].

Finally, given a line bundle $\pi : \mathbf{E} \to \mathbf{N}$ and a curve with cusps and normal crossings $\Gamma \subset \mathbf{N}$, we ask whether a generic surface (i.e. $\pi_{|S}$ is excellent) $S \subset \mathbf{E}$, having Γ as apparent contour, exists. Since we can construct all excellent extensions of the given curve, the problem reduce to find a factorization $F = \pi \circ F_1$ of a given excellent map F, asking for $F_1 : S \to \mathbf{E}$ to be an embedding. In §2 we use the methods developed in §1 to find combinatorial, necessary and sufficient conditions in order that F possesses such a factorization (Theorem 2.32).

0. Immersions with assigned boundary

Definition 0.1. Let k be a positive integer, we call generic k-curve an immersion $f : \coprod_{h=1}^{k} \mathbf{S}^1 \to \mathbf{N}$, whose image $\Gamma = \coprod_{h=1}^{k} \Gamma_h$ is smooth up to a finite number of normal crossings.

Definition 0.2. Let S be a surface (not necessarily connected) with boundary and f a generic k-curve, we say that $F : S \to \mathbf{N}$ extends f if and only if F is an immersion, $\partial S = \coprod_{h=1}^{k} \mathbf{S}^1$ and $F|_{\partial S} = f$. We denote by $E(f)$ the set of all extensions of f. Fixed a point $\infty \in \mathbf{N} - \Gamma$, we denote by $E^{\beta}_{\infty}(f)$ the set of all extensions such that $\#F^{-1}(\infty) = \beta$.

If f is a generic k-curve, then the bundle $f^*T\mathbf{N} \to \coprod_{h=1}^{k} \mathbf{S}^1$ (i.e. the pull-back via f of the tangent bundle to $\mathbf{N}$) has a canonical trivial sub-bundle, τf, spanned by the never zero cross section f'. Let $\nu f = f^*T\mathbf{N}/_{\tau f}$. The two sub-bundle τf and νf are called the tangent and normal bundles to f.

Definition 0.3. We say that f is sided if ν_f is the trivial bundle. A side for f is a never zero cross section of νf, up to multiplication by a positive function.

REMARK 0.4. Since νf can be realized as a sub-bundle of $f^*T\mathbf{N}$, in such a way that $f^*T\mathbf{N} = \tau f \oplus \nu f$, f is sided if and only if there exists a vector field along f, which is transversal to f.

A notion of rotation number can be given for a sided curve in a surface, with respect to a fixed vector field X on $\mathbf{N}$, not vanishing on Γ. We denote by $R_X(f)$ the rotation number of the sided curve f with respect to the vector field X. We do not give its definition here (for a definition see [20], [4], [13]), we only remark that this number essentially counts how many times a vector specifying the side turns with respect to X and that it coincide with the usual rotation number for plane curves, endowing such a curve with the side induced by the orientations of $\mathbb{R}^2$ and taking X to be a constant (never zero) vector field.

Let S be a compact surface with boundary and let $F : S \to \mathbf{N}$ be a generic immersion (i.e. $F|_{\partial S}$ is a generic curve), then $F|_{\partial S}$ has a canonical side defined by the image of an inward pointing vector field on ∂S. The following fact holds (see [4], [13]):

PROPOSITION 0.5. *Let $F : S \to \mathbf{N}$ be a generic immersion and let X be a vector field having at most one isolated zero at the point $\infty \notin \Gamma$. Then*

$$R_X\big(F|_{\partial S}\big) = \chi(S) - \beta\chi(\mathbf{N}). \tag{0-1}$$

Here χ denotes Euler's characteristic and $\beta = \#F^{-1}(\infty)$.

Let $f : \coprod_{h=1}^{k} \mathbf{S}^1 \to \mathbf{N}$ be a sided curve and $\infty \notin \Gamma$ a fixed point.

DEFINITION 0.6. A set of segments for the curve f is a set $R = \{r_i\} = A \cup \Omega$, $A = \{\alpha_i\}$ and $\Omega = \{\omega_j\}$, such that:

(1) each α_i is an oriented, smooth arc, diffeomorphic to a segment of the real line, starting at a point in some component of $\mathbf{N} - \Gamma$ and ending to ∞;
(2) the ω_j's are smooth (diffeomorphic to $\mathbf{S}^1$) representatives of a minimal system of generators of $\pi_1(\mathbf{N}, \infty)$;
(3) each r_i is in general position with respect to Γ (i.e. misses all crossings and is transversal to Γ);
(4) for all $i \neq j \quad r_i \cap r_j = \infty$.

We say that R is a system of segments if and only if the following holds too:

(5) if C is a component of $\mathbf{N} - \Gamma$ and $\infty \notin C$, then some α_i starts from C.

Given a set of segments for the k-curve f, fix a neighborhood, U_∞, of the point ∞, such that $U_\infty \cap \Gamma = \varnothing$. For each $r \in R$ fix an orientable neighborhood U_{r_i} of $r - U_\infty$ and an orientation on it. Label each point in $R \cap \Gamma$ by a letter $x^e_{i,j,\mu}$, being i, j integers and $e, \mu = \pm 1$, according to the following rules (see Figure 1):

(1) $x^e_{i,j,\mu} \in r_i$;
(2) $e = 1$ if the segment r_i crosses Γ from left to right (positive point), with respect to the side of f; $e = -1$ otherwise (negative point);
(3) $\mu = 1$ if Γ crosses r_i from left to right, with respect to the orientation of U_{r_i}; $\mu = -1$ otherwise;
(4) for all points $x^e_{i,j,\mu}, x^e_{i,j',\mu'} \in r_i$, $j < j'$ if and only if $x^e_{i,j,\mu} < x^e_{i,j',\mu'}$ with respect to the ordering induced by the orientation on r_i.

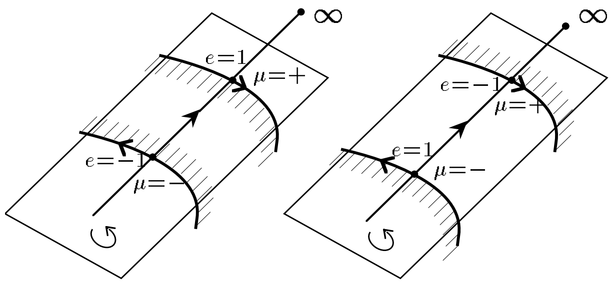

FIGURE 1

Finally, define a set of words $w = \{w_1, \dots, w_k\}$ by the following construction: fix points $p_h \in \Gamma_h$, being neither crossings of Γ nor points in $R \cap \Gamma$; for each h, walk along Γ_h, starting from p_h and following its orientation, and write down all the labels you meat, until you come back to p_h. Call w_h the word obtained this way.

DEFINITION 0.7. The set $w = \{w_1, \dots, w_k\}$ is called the k-word of the generic k-curve f, with respect to the set of segments R.

REMARK 0.8. Clearly, w depends on the choice of the base points p_h. Indeed, if we change the base point p_h, then the word w_h changes by the action of a cyclic permutation. We shall always consider words up to cyclic permutations.

DEFINITION 0.9. We call 0-assemblage for the k-word w, a set $\mathfrak{A}$, whose elements are unordered pairs $\{x^{-1}_{i,j,\mu}, x_{i,j',\mu'}\}$ of letters in w, such that:

(1) if $\{x^{-1}_{i,j,\mu}, x_{i,j',\mu'}\} \in \mathfrak{A}$, then $j < j'$;
(2) each letter, corresponding to a point in $\Omega \cap \Gamma$, appears in some pair of $\mathfrak{A}$;
(3) each negative letter (i.e. $e = -1$) appears in some pair of $\mathfrak{A}$;
(4) each letter appears at most in one pair of $\mathfrak{A}$.

Given f and R as above, fix a neighborhood V_∞ of ∞ and a diffeomorphism $\Phi_\infty : V_\infty \to \mathbf{B}^2$ (here $\mathbf{B}^2$ denotes the unitary open ball in $\mathbb{R}^2$), such that:

(1) $\Gamma \cap V_\infty = \varnothing$;
(2) $\Phi_\infty(R \cap V_\infty)$ is a set of rays starting at the origin;

and for each integer $l \geq 1$, consider the curve $g_l : \mathbf{S}^1 \to \mathbf{N}$, $g_l(t) = \Phi^{-1}_\infty\big((\frac{1}{3} + \frac{1}{3l})e^{it}\big)$, endowed with the side pointing outside V_∞.

DEFINITION 0.10. We call β-expansion of the k-curve f, the $(k+\beta)$-curve f^β, obtained adding $g_1, \dots, g_\beta$ to f. If w is a word for f, we call β-expansion of the k-word w, the $(k+\beta)$-word w^β for f^β defined by the same set of segments as w. Finally, we call β-assemblage for the word w a 0-assemblage for w^β.

REMARK 0.11. $w^\beta = w \cup \{u_1, \dots, u_\beta\}$, u_l being the word for g_l. Observe that the words $u_1, \dots, u_\beta$ are equals up to shift the index j of all letters. We define an action of the symmetric group $\mathfrak{S}_\beta$ on $\mathcal{A}^\beta(w)$, $\sigma\mathfrak{A}$ being the set obtained from $\mathfrak{A}$ exchanging each letter in u_l with the corresponding one in $u_{\sigma l}$. Clearly $\sigma\mathfrak{A}$ fulfills conditions (1)–(4) in Definition 0.9, hence it is actually a β-assemblage for w.

A β-assemblage, $\mathfrak{A}$, defines a graph, $G(\mathfrak{A})$, as follows: take $k+\beta$ disjoint, oriented circles $C_1, \dots, C_k$ and $C_{k+1}, \dots, C_{k+\beta}$, on each circle C_h, $h = 1, \dots, k$ C_{k+l}, $l = 1, \dots, \beta]$ choose points corresponding to the letters of the word w_h $[u_l]$, ordered as in w_h $[u_l]$, and join by an extra edge (we call such an edge a proper edge) all pairs of points corresponding to pairs of letters in $\mathfrak{A}$. Finally weight each proper edge by -1 or 1 according as the μ signs of its vertices agree or not.

A graph as just described (i.e. all vertices lay on k disjoint, oriented circles and all proper edges are weighted by ± 1) will be called a weighted k-circular graph. We denote by $l(G)$ the number of connected components of G. If $\mathfrak{A}$ is a β-assemblage, then $l(\mathfrak{A})$ will denote the number of components of its associated graph.

Definition 0.12. Let G be a connected weighted k-circular graph and S a surface with k boundary component. We say that G is embeddable in S if and only if there exists a map $\varphi : G \to S$ such that the following three hold:

(1) φ is an homeomorphisms onto its image;
(2) $\varphi(\bigcup_h C_h) = \partial S$;

for each proper edge ℓ of G, let U_ℓ be a neighborhood of $\varphi(\ell)$ homeomorphic to $\ell \times [-1, 1]$. Two semiorientations are naturally defined on $\partial S \cap U_\ell$: the one induced as boundary of U_ℓ and the one induced from the orientation of the circles C_h.

(3) the just described semiorientations on $\partial S \cap U_\ell$ agree if and only if the weight of the edge ℓ is equal to 1.

Obviously, every weighted k-circular graph is embeddable in some surface.

Definition 0.13. If G is a connected, weighted k-circular graph we call genus of G the number $g(G) = \min\{g(S) \mid G \text{ is embeddable in } S\}$.

It is easy to prove the following (see [13]):

Proposition 0.14. *Let G' be obtained from G by reversing the orientation of a circle and changing the weight of all proper edges having exactly one vertex on that circle, then $g(G') = g(G)$. If G is embeddable in S and $g(G) = g(S)$, then S is orientable if and only if there exists a finite sequence $G = G_0, \ldots, G_n$ of graphs such that for all i, G_{i+1} is obtained from G_i as above, and all proper edges of G_n are positively weighted, in this case we say G is positive.*

The second part of the previous statement allows us to define another noteworthy number associated to the graph:

Definition 0.15. We call weighted genus of the connected k-circular graph G the number $h(G) = 2g(G)$ or $g(G)$ according as G is positive or not. If G is not connected and G_i are its connected component, we call weighted genus of G the number $h(G) = \sum_i h(G_i)$. We call weighted genus of a β-assemblage, the weighted genus of its associated graph and we denote it by $h(\mathfrak{A})$.

The following proposition holds:

Proposition 0.16. *Let $f : \coprod_{h=1}^{k} \mathbf{S}^1 \to \mathbf{N}$ be a generic curve, and w be the word for f defined by a system of segments, then for all $\mathfrak{A} \in \mathcal{A}^\beta(w)$*

$$R_X(f) \geq 2l(\mathfrak{A}) - k - h(\mathfrak{A}) - \beta\chi(\mathbf{N}). \tag{0-2}$$

DEFINITION 0.17. Let w be defined by a system of segments. A β-assemblage $\mathfrak{A} \in \mathcal{A}^{\beta}(w)$ is said to be minimal if and only if in (0-2) equality holds. We denote by $\mathcal{A}^{\beta}_{m}(w)$ the set of minimal β-assemblages of w.

It is not hard to see that if $\sigma \in \mathfrak{S}_{\beta}$, $\mathfrak{A} \in \mathcal{A}^{\beta}_{m}(w)$, then $\sigma\mathfrak{A} \in \mathcal{A}^{\beta}_{m}(w)$, hence $\mathfrak{S}_{\beta}$ acts on $\mathcal{A}^{\beta}_{m}(w)$. Denote right equivalence by $\overset{\mathrm{r}}{\sim}$, the following theorem holds:

THEOREM 0.18. *Let $f : \coprod_{h=1}^{k} \mathbf{S}^1 \to \mathbf{N}$ be a generic k-curve and let w be the k-word for f defined by a system of segments. Then there exists a bijection between the set $E^{\beta}_{\infty}(f)/_{\overset{\mathrm{r}}{\sim}}$ and the set $\mathcal{A}^{\beta}_{m}(w)/_{\mathfrak{S}_{\beta}}$.*

SKETCH OF PROOF. We only show how to define such a bijection. First suppose $\beta = 0$. Let $F \in E^{0}_{\infty}(w)$ be fixed. The set $\widetilde{R} = F^{-1}(R)$ is made up by smooth arcs in S, which are oriented by the immersion. Label each point of $\widetilde{R} \cap \partial S$ by the same label as the corresponding point in Γ, and observe that each positive labeled point (i.e. $e = 1$) is the ending point of some arc, while each negative labeled point (i.e. $e = -1$) is the starting point of some arc. Furthermore, since $F^{-1}(\infty) = \varnothing$, every arc must end to ∂S. Pairing the letters corresponding to the vertices of those arcs in $\widetilde{R}$, that have both vertices in ∂S, we actually obtain a minimal 0-assemblage, $\mathfrak{A}(F)$, and the mapping $F \mapsto \mathfrak{A}(F)$ is a bijection $E^{0}_{\infty}(f)/_{\overset{\mathrm{r}}{\sim}} \to \mathcal{A}^{0}_{m}(w)$.

If $F \in E^{\beta}_{\infty}(f)$, let $D_1, \ldots, D_{\beta} \subset \mathbf{N}$ be the disks bounding the curves g_l that define the β-expansion of f (see Definition 0.10), call $\{p_1, \ldots, p_{\beta}\} = F^{-1}(\infty)$ and U^{j}_{i} the interior of the connected component of $F^{-1}(D_i)$ containing p_j. For each $\sigma \in \mathfrak{S}_{\beta}$, let $S_{\sigma} = S - \bigcup_i U^{\sigma i}_{i}$. Clearly $F_{\sigma} = F|_{S_{\sigma}} \in E^{0}_{\infty}(f^{\beta})$, hence, by the case $\beta = 0$, we obtain an assemblage $\mathfrak{A}(F_{\sigma}) \in \mathcal{A}^{0}_{m}(w^{\beta}) = \mathcal{A}^{\beta}_{m}(w)$. Such a construction actually defines a bijection between $E^{\beta}_{\infty}(f)/_{\overset{\mathrm{r}}{\sim}}$ and $\mathcal{A}^{\beta}_{m}(w)/_{\mathfrak{S}_{\beta}}$. $\square$

The statement of Theorem 0.18 can be a little improved.

DEFINITION 0.19. A connected component, say C, of $\mathbf{N} - \Gamma$ is said to be negative if the side of the curve points outward C at all points in ∂C; otherwise it is said to be positive. A set of segments is called a reduced system of segment if:

(5′) at least one of the α_i's starts from each negative component.

REMARK 0.20. Proposition 0.16, Definition 0.17 and finally Theorem 0.18 can be restated and proved (see [13]), replacing the words "system of segments" by "reduced system of segments". These improved forms of such statements will be used in §1.

Now we prove a lemma, that we shall use in §1. We need one more definition:

DEFINITION 0.21. We say that a generic k-curve has a curl if there exist $t_0, t_1 \in \mathbf{S}^1$ such that $f(t_0) = f(t_1)$ and $f|_{[t_0,t_1]}$ bounds a disk, $D \subset \mathbf{N}$, such that $D \cap \Gamma = \partial D$. We say that the curl is positive or negative according as $D - \partial D$ is a positive or negative component.

LEMMA 0.22. *Suppose that the k-curve f has a positive curl, then for every word w defined by a reduced system of segments, $\mathcal{A}^{\beta}_{m}(w) = \varnothing$.*

PROOF. It is enough to prove that $\mathcal{A}^{\beta}_{m}(w) = \varnothing$ for the word defined by some reduced system of segments. Let D be the disk bounded by the curl and observe

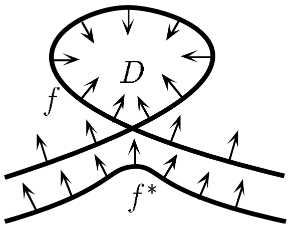

FIGURE 2

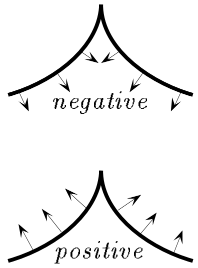

FIGURE 3

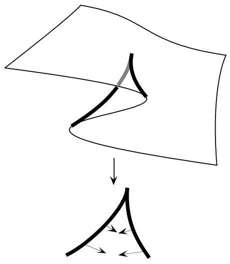

FIGURE 4

that we can construct a reduced system of segment R such that $R \cap D = \emptyset$. Indeed, fix $\infty \notin D$, since the curl is positive, there is no need to draw segments starting from D and all other segments can be drawn far from D. Let f^* be the curve obtained removing the curl as suggested in Figure 2 and let w and w^* the words defined by R for f and f^*. Clearly, R is a reduced system of segments for f^* too and since $R \cap D = \emptyset$, $w = w^*$. Applying (0-2) to f^* and using the fact that $R_X(f) = R_X(f^*) + 1$, we obtain: $R_X(f) > 2l(\mathfrak{A}) - k - h(\mathfrak{A}) - \beta\chi(\mathbf{N})$ for all $\mathfrak{A} \in \mathcal{A}^\beta(w)$. $\square$

1. Extending curves with cusps and normal crossings

Let I be an open interval of the real line $\mathbb{R}$, and $f : I \to \mathbf{N}$ be a curve.

DEFINITION 1.1. A point $t_0 \in I$ is called a cusp point if and only if there exist germs of diffeomorphism $\gamma : (I, t_0) \to (\mathbb{R}, 0)$ and $\varphi : (\mathbf{N}, f(t_0) \to (\mathbb{R}^2, 0)$ such that $\varphi \circ f \circ \gamma^{-1}(s) = (s^2, s^3)$. We say that $f : \coprod_{h=1}^k \mathbf{S}^1 \to \mathbf{N}$ is a k-curve with cusp and normal crossing (briefly a CN k-curve) if $f'(t) \neq 0$, except for a finite number of cusp points, and f is injective except for a finite number of normal crossings.

Let $\mathcal{C}$ be the set of cusp points of the CN k-curve $f : \coprod_{h=1}^k \mathbf{S}^1 \to \mathbf{N}$.

DEFINITION 1.2. A side for $f : \coprod_{h=1}^k \mathbf{S}^1 \to \mathbf{N}$ is a side for $f|_{\coprod_{h=1}^k \mathbf{S}^1 - \mathcal{C}}$, that, in a neighborhood of each cusp, is directed either inward or outward the cusp on both branches of the cusp itself. A cusp is said to be positive or negative according as the side is outward or inward pointing (see Figure 3). Finally, we say that a side for f is coherent if and only if all cusps are negative.

Let $F : S \to \mathbf{N}$ be an excellent mapping, by definition $F|_{\Sigma_F}$ is a CN curve.

PROPOSITION 1.3. *The curve $F|_{\Sigma_F}$ has a coherent side.*

PROOF. Take the folding side of the map (see Figure 4 and [18]). $\square$

Local extensions. Let $f : \mathbf{S}^1 \to \mathbf{N}$ be a sided CN curve and let M be either a cylinder or a Möbius band (i.e. either $M = \mathbf{S}^1 \times [-1, 1]$ or $M = \mathbb{R} \times [-1, 1]/_{(x,t)=(x+1,-t)}$) and identify $\mathbf{S}^1$ with the central circle in M (i.e. either $\mathbf{S}^1 \times \{0\}$ or $\mathbf{S}^1 = \mathbb{R} \times \{0\}/_{(x,0)=(x+1,0)}$). By a local excellent extension of f we shall mean an excellent mapping $F : M \to \mathbf{N}$ such that $\Sigma_F = \mathbf{S}^1$, $F|_{\mathbf{S}^1} = f$ and inducing the given side on f. If $f : \coprod_{h=1}^k \mathbf{S}^1 \to \mathbf{N}$ is a sided CN k-curve a local

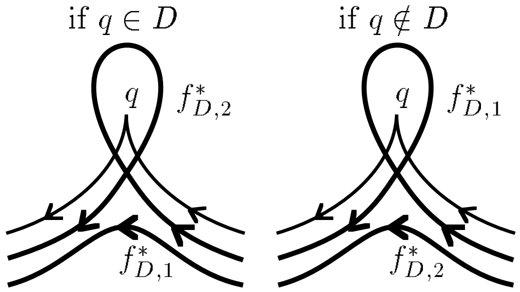

FIGURE 5

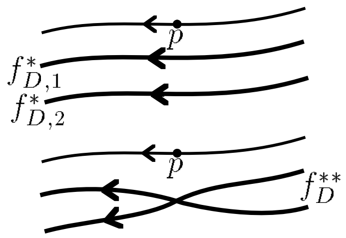

FIGURE 6

excellent of f will mean a mapping $F : \coprod_{h=1}^k M_h \to \mathbf{N}$ such that $F|_{M_h}$ is a local excellent extension of f restricted to the h^{th} circle.

Let $f : \mathbf{S}^1 \to \mathbf{N}$ be a sided CN curve and let $D \subseteq \mathcal{C}$.

DEFINITION 1.4. We call D-deformation of the first kind of f the generic 2-curve $f^*_D = f^*_{D,1} \amalg f^*_{D,2}$, that one obtains by doubling f and deforming its cusps as suggested in Figure 5. Fix a point $p \in \Gamma$ being neither a crossing nor a cusp, we call D-deformation of the second kind the generic 1-curve f^{**}_D obtained modifying f^*_D in a neighborhood of p as suggested in Figure 6.

REMARK 1.5. It is clear that $f^*_D = f^*_{\mathcal{C}-D}$ and $f^{**}_D = f^{**}_{\mathcal{C}-D}$. By local arguments, it is easy to see that if $F : M \to \mathbf{N}$ is a local excellent extension of f, then there exists a tubular neighborhood U of $\mathbf{S}^1$ such that either $f|_{\partial U} = f^*_D$ or $F|_{\partial U} = f^{**}_D$ (according as M is either a cylinder or a Möbius band) for some $D \subseteq \mathcal{C}$. This set D is unique up to complementation. We call such a D, up to complementation, the deformation set induced by F.

Let $f : \coprod_{h=1}^k \mathbf{S}^1 \to \mathbf{N}$ be a sided CN k-curve, D as above and let $H \subseteq \{1, \dots, k\}$; a pair (D, H) will be called a deformation pair. Denote by $\bar{H}$ the complement of H and let k_H be the number $k_H = 2\#H + \#\bar{H}$.

DEFINITION 1.6. We call (D, H)-deformation of f the generic k_H-curve defined by $f_{(D,H)} = \{(f_h)^*_D \mid h \in H\} \amalg \{(f_h)^{**}_D \mid h \in \bar{H}, \}$.

REMARK 1.7. Let $\mathcal{C}_h$ be the set of cusp points in the h^{th} circle and $D_h = D \cap \mathcal{C}_h$, if D' is such that either $D'_h = D_h$ or $D'_h = \mathcal{C}_h - D_h$ for all h, then $f_{(D,H)} = f_{(D',H)}$. If F is a local excellent extension of f, then there exists a tubular neighborhood U of $\coprod_{h=1}^k \mathbf{S}^1$ such that $F|_{\partial U} = f_{(D,H)}$ for some pair (D, H). Furthermore if (D, H) and (D', H') are two different such pairs, then $H = H'$ and D, D' are as above. We call such a pair (D, H), up to this relation, the deformation pair induced by F.

REMARK 1.8. Let $F : M \to \mathbf{N}$ be a local excellent extension of $f : \mathbf{S}^1 \to \mathbf{N}$ and let t_0 be a cusp point. If $v \in T_{t_0}M - \ker\bigl(dF(t_0)\bigr)$, then $dF(t_0)[v]$ is a vector tangent to the cusp. Identify M with the quotient space of $\mathbb{R} \times [-1, 1]$ by the usual relations in such a way that $\pi^{-1}(p) = \mathbb{Z}$, where p is the fixed point in Γ and π is the quotient map. Let $\widetilde{F} = F \circ \pi$. It can be easily shown that the deformation set induced by F is given, up to complementation, by $D = \bigl\{t \in (0, 1) \bigm| (t, 0) \text{ is a cusp point and } d\widetilde{F}(t, 0)\bigl[(0, 1)\bigr] \text{ points inward the cusp}\bigr\}$.

Using this characterization, it is not hard to prove the following proposition:

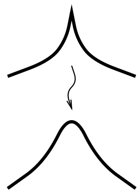

FIGURE 7

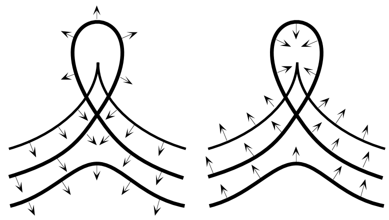

FIGURE 8

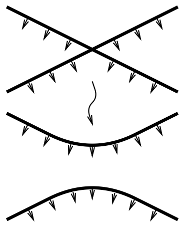

FIGURE 9

PROPOSITION 1.9. *Let $f : \coprod_{h=1}^{k} \mathbf{S}^1 \to \mathbf{N}$ be a CN k-curve endowed with e coherent side, then for all deformation pairs (D, H), there exists a local excellent extension of f inducing (D, H).*

Let $F, G : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be two generic, singular germs having the same apparent contour (i.e. $F(\Sigma_F) = G(\Sigma_G) = \Gamma$), then they have the same normal form with respect to left-right equivalence, say *ii)* or *iii)* in introduction. Standard arguments in singularity theory, prove the following lemma.

LEMMA 1.10. *Let F, G be as above and suppose they induce the same side, then there exists a germ of diffeomorphism Φ such that $F = G \circ \Phi$. If their normal form is iii) such a germ Φ is uniquely determined. If their normal form is ii) there are exactly two such germs, one of which is orientation preserving and the other one is orientation reversing.*

PROPOSITION 1.11. *Let $f : \coprod_{h=1}^{k} \mathbf{S}^1 \to \mathbf{N}$ be a CN k-curve and F, G two local, excellent extensions of f. F and G are locally (around $\coprod_{h=1}^{k} \mathbf{S}^1$) right equivalent if and only if they induce the same deformation pair.*

PROOF. If $F \overset{r}{\sim} G$, they obviously induce the same deformation pair. Conversely, suppose that F and G induce the same pair. Clearly, it is enough to prove the thesis for a CN 1-curve. Let M be the domain of F and G. By the previous lemma we can find finite open covers $\{U_i\}_{i=1,\ldots,n}$, $\{V_i\}_{i=1,\ldots,n}$ of $\mathbf{S}^1$ and diffeomorphisms $\Phi_i : U_i \to V_i$ such that $F|_{U_i} = G|_{V_i} \circ \Phi_i$. Lifting all maps to the universal covering of M, we see that, since F and G induce the same deformation set, all the uniquely determined diffeomorphisms are orientation preserving [reversing]. To end the proof it is enough to choose all the other ones to be orientation preserving [reversing] and paste them together. $\square$

Rotation number. Let $f : \coprod_{h=1}^{k} \mathbf{S}^1 \to \mathbf{N}$ be a sided CN k-curve, and let X be a vector field on $\mathbf{N}$ with no zeros on Γ. Denote by f^* the generic k-curve obtained by deforming all cusps in the way suggested in Figure 7. With the notations introduced above, $f^* = \coprod_{h=1}^{k} (f_h)^*_{\mathcal{C},1}$. Since a deformation of f has a side, canonically induced by the side of f (Figure 8), we can give the following definition:

DEFINITION 1.12. We call rotation number of f with respect to X the number $R_X(f) = R_X(f^*)$, where f^* is endowed with the side induced by f.

LEMMA 1.13. *Let $f : \mathbf{S}^1 \to \mathbf{N}$ be a sided, CN 1-curve and let $D \subseteq \mathcal{C}$, then $R_X(f^{**}_D) = R_X(f^*_D) = 2R_X(f) - c^- + c^+$, where c^+ and c^- denote respectively the number of positive and negative cusps.*

Proof. Since rotation number does not change modifying a curve as suggested in Figure 9, the left hand side equality holds. Denote by $c_D^{\pm}$ the number of positive and negative cusps in D. $f^*_{D,1}$ $[f^*_{D,2}]$ is obtained adding $c^+_{\mathcal{C}-D}$ $[c^+_D]$ positive curls and $c^-_{\mathcal{C}-D}$ $[c^-_D]$ negative curls to f^*, hence $R_X(f^*_{D,1}) = R_X(f^*) + c^+_{\mathcal{C}-D} - c^-_{\mathcal{C}-D}$ and $R_X(f^*_{D,2}) = R_X(f^*) + c^+_D - c^-_D$. Sum these two equalities and obtain the thesis. □

Lemma 1.14. *Let $f : \coprod_{h=1}^k \mathbf{S}^1 \to \mathbf{N}$ be a CN k-curve and (D,H) a deformation pair, then $R_X\big(f_{(D,H)}\big) = 2R_X(f) - c^- + c^+$.*

Proof. Follows immediately by the previous lemma. □

From now on, we shall suppose to have fixed a point $\infty \notin \Gamma$ and a vector field X vanishing at most at ∞.

Proposition 1.15. *Let $F : S \to \mathbf{N}$ be an excellent map, then:*

$$2R_X\big(F|_{\Sigma_F}\big) - c = \chi(S) - \beta\chi(\mathbf{N}). \tag{1-1}$$

Where c denotes the number of cusps, and $\beta = \#F^{-1}(\infty)$.

Proof. Put $f = F|_{\Sigma_F}$. By Proposition 1.3, all cusps are negative, hence, by Lemma 1.14, $R_X\big(f_{(D,H)}\big) = 2R(f) - c$, for all deformation pairs (D,H). Let U be a tubular neighborhood of Σ_F, such that $F|_{\partial U} = f_{(D,H)}$. The two sides on $f_{(D,H)}$, induced respectively by the immersion $F|_{S-U}$ and by the side of f, are clearly the same, hence, by (0-1), $R_X\big(f_{(D,H)}\big) = \chi(S-U) - \beta\chi(\mathbf{N})$. To end the proof, observe that $\chi(S) = \chi(S-U)$. □

Remark 1.16. The previous proposition gives one more condition in order that a curve may be the apparent contour of an excellent map. Nevertheless it is not hard to construct examples of curves fulfilling (1-1) for some choice of a surface S, but being the apparent contour of no excellent map $S \to \mathbf{N}$. We remark also that (1-1) implies a generalization of a classical theorem of Thom [21], asserting that $\chi(S) \equiv c \mod 2$ for all excellent maps $F : S \to \mathbb{R}^2$. Indeed, denoting by $\deg_2$ the modulo two degree, we have:

Theorem 1.17. *Let $F : S \to \mathbf{N}$ be an excellent map, then $\chi(S) \equiv c \mod 2$, if and only if either $\chi(\mathbf{N})$ is even or $\deg_2(F) = 0$.*

Words. Let $f : \coprod_{h=1}^k \mathbf{S}^1 \to \mathbf{N}$ be a CN k-curve and $\infty \in \mathbf{N} - \Gamma$ a fixed point.

Definition 1.18. A set of segments for f is a set $R = A \cup B \cup \Omega$ of smooth, oriented arcs in $\mathbf{N}$, such that:

(1) each $r \in A \cup B$ is diffeomorphic to a closed interval of the real line. The ending point of such arcs being ∞ and the starting point being either a point in a connected component of $\mathbf{N} - \Gamma$, if $r \in A$, or a cusp, if $r \in B$;
(2) Ω is a set of smooth (diffeomorphic to $\mathbf{S}^1$) representatives of a minimal set of generators of $\pi_1(\mathbf{N}, \infty)$;
(3) $r \cap r' = \infty$ for all $r, r' \in R$, $r \neq r'$;
(4) each $r \in R$ is in general position with respect to Γ, that is r contains neither crossings nor cusps (except at most the starting point), is transversal to Γ and if it starts from a cusp, it points outward the cusp.

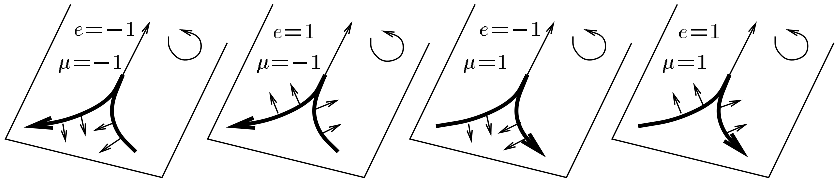

FIGURE 10

We say that R is a system of segments for f if the following holds too:

(5) at least one segment starts from each component of $\mathbf{N}-\Gamma$ not containing ∞ and from each cusp.

DEFINITION 1.19. Suppose f is sided, and let $r \in R$. We say $p \in R \cap \Gamma$ is positive if and only if either p is not a cusp and r crosses Γ from the left to the right or p is a positive cusp; otherwise we say p is negative.

Fix a neighborhood U_∞ of ∞ not intersecting Γ and oriented neighborhoods V_r of $r - U_\infty$ for all $r \in R$. Next put labels $x^e_{i,j,\mu}$ on each point in $R \cap \Gamma$ according to the same rules as in §1, except for cusp points where the index μ is $+1$ or -1 according as the curve crosses the segment from left to right or from right to left (see Figure 10). A set of k words can be associated to f as in §1.

DEFINITION 1.20. We call the set $w = \{w_1, \dots, w_k\}$ obtained this way, the k-word of f, with respect to the set of segments R.

REMARK 1.21. As for generic curves (Remark 0.8), the word of a CN curve will be considered up to the action of cyclic permutations.

Let $f : \mathbf{S}^1 \to \mathbf{N}$ be a CN curve, w a word for f and $D \subseteq \mathcal{C}$.

DEFINITION 1.22. We call D-deformations of the first and second kind of w respectively the words $w^*_D = \{w^*_{D,1}, w^*_{D,2}\}$ and $w^{**}_D = w^*_{D,1} w^*_{D,2}$, where $w^*_{D,1}$ is obtained from w by erasing all letters corresponding to cusps in D and $w^*_{D,2}$ by erasing the ones corresponding to cusps in $\mathcal{C} - D$.

Let f be a CN k-curve, w a k-word for f and (D, H) a deformation pair.

DEFINITION 1.23. The (D, H)-deformation of w is the k_H-word $w_{(D,H)} = \big\{(w_h)^*_D \bigm| h \in H\big\} \amalg \big\{(w_h)^{**}_D \bigm| h \in \bar{H}\big\}$, where $\bar{H}$ and k_H are as in Definition 1.6.

DEFINITION 1.24. A β-assemblage for w is a pair $\mathbf{A} = \big((D, H), \mathfrak{A}\big)$, where (D, H) is a deformation pair and $\mathfrak{A}$ is a β-assemblage for $w_{(D,H)}$ (as defined in Definition 0.10). We denote by $\mathcal{A}^\beta(w)$ the set of β-assemblages of w. If $\mathbf{A} = \big((D, H), \mathfrak{A}\big) \in \mathcal{A}^\beta(w)$, we call number of components and weighted genus of $\mathbf{A}$ respectively the numbers $l(\mathbf{A}) = l(\mathfrak{A})$ and $h(\mathbf{A}) = h(\mathfrak{A})$.

REMARK 1.25. All letters, except the ones corresponding to cusps, appear twice in the deformed word $w_{(D,H)}$. In the previous definition, the two copies of the same letter, must be considered as different ones.

PROPOSITION 1.26. *Let R be a system of segments for f, then R is a reduced system of segments for $f_{(D,H)}$.*

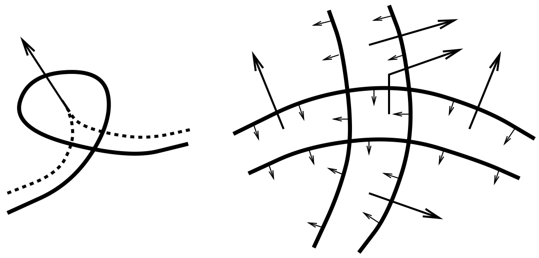

FIGURE 11

PROOF. Clearly, R is a set of segments for $f_{(D,H)}$, (the deformation can be done in such a way that R is a set of segments for $f_{(D,H)}$). Each connected components of $\mathbf{N}-\Gamma_{(D,H)}$ either is essentially a component of $\mathbf{N}-\Gamma$ or is generated by the deformation. At least one segment starts from each component of the first kind. The new ones are bounded by either the curl around a cusp or parallel branches of $\Gamma_{(D,H)}$. In the first case, the segment, starting from the corresponding cusp, starts from the new component while in the second case, it is easy to see that such components are positive (see Figure 11), hence condition $(5')$ of Definition 0.19 holds. $\square$

REMARK 1.27. Let $\widetilde{w}_{(D,H)}$ the word for $f_{(D,H)}$ defined by R, clearly $\widetilde{w}_{(D,H)}$ is obtained renumbering the second index of all letters in $w_{(D,H)}$ in such a way to preserve the previous ordering of letters belonging to the same segment; hence a β-assemblage $\mathbf{A}=\big((D,H),\mathfrak{A}\big)$ for w defines naturally a β-assemblage $\widetilde{\mathfrak{A}}_{\mathbf{A}}$ for $\widetilde{w}_{(D,H)}$. Furthermore, since letters having the same indexes in $w_{(D,H)}$ have also the same exponent, the corresponding letters in $\widetilde{w}_{(D,H)}$ cannot be paired by any β-assemblage. This argument proves the following:

PROPOSITION 1.28. *The mapping $\mathbf{A}\mapsto\widetilde{\mathfrak{A}}_{\mathbf{A}}$ is a bijection between the set of β-assemblages of w having (D,H) as deformation pair and $\mathcal{A}^{\beta}\big(\widetilde{w}_{(D,H)}\big)$. Furthermore $\mathbf{A}$ and $\widetilde{\mathfrak{A}}_{\mathbf{A}}$ have the same number of components and weighted genus.*

PROPOSITION 1.29. *Let f be a CN k-curve and w the word defined by a system of segments. Then for all $\mathbf{A}\in\mathcal{A}^{\beta}(w)$,*

$$2R_X(f)-c_-+c_+\geq 2l(\mathbf{A})-k_H-h(\mathbf{A})-\beta\chi(\mathbf{N}). \tag{1-2}$$

PROOF. Use Proposition 1.28, (0-2) and Lemma 1.14. $\square$

DEFINITION 1.30. Let w be the k-word for the CN k-curve f, defined by a system of segments. $\mathbf{A}\in\mathcal{A}^{\beta}(w)$ will be called minimal if in (1-2) equality holds. We shall denote by $\mathcal{A}^{\beta}_{m}(w)$ the set of minimal β-assemblages of w.

REMARK 1.31. The mapping $\mathbf{A}\mapsto\widetilde{\mathfrak{A}}_{\mathbf{A}}$ is a bijection between the set of minimal β-assemblages of w, having (D,H) as deformation pair, and $\mathcal{A}^{\beta}_{m}\big(\widetilde{w}_{(D,H)}\big)$.

Let f be a CN k-curve, and suppose that the h^{th} component of f has no cusps. Let (D,H) be a deformation pair, then $(w_h)^*_{D,1}=(w_h)^*_{D,2}=w_h$. We define an involution ι_h on the set of letters of $w_{(D,h)}$ by mapping each letter of $(w_h)^*_{D,1}$ to the corresponding one of $(w_h)^*_{D,2}$ and leaving fixed all other letters.

Let $\mathbf{A} = \big((D,H),\mathfrak{A}\big)$ be a β-assemblage of w, we denote by $\iota_h\mathbf{A}$ the β-assemblage $\big((D,H),\iota_h\mathfrak{A}\big)$, where $\iota_h\mathfrak{A} = \big\{\{\iota_h(x^{-1}_{i,j,\mu}),\iota_h(x_{i,j',\mu'})\} \bigm| \{x^{-1}_{i,j,\mu},x_{i,j',\mu'}\}\in\mathfrak{A}\big\}$.

Definition 1.32. We say that $\mathbf{A}$ is ι-equivalent to $\mathbf{A}'$ (in symbols $\mathbf{A} \overset{\iota}{\sim} \mathbf{A}'$), if and only if there exist involutions $\iota_{h_1},\dots,\iota_{h_n}$ such that $\mathbf{A}' = \iota_{h_n}\dots\iota_{h_1}\mathbf{A}$. We say that $\mathbf{A} = \big((D,H),\mathfrak{A}\big)$ and $\mathbf{A}' = \big((D',H'),\mathfrak{A}'\big)$ are δ-equivalent (in symbols $\mathbf{A} \overset{\delta}{\sim} \mathbf{A}'$) if the two deformation pairs are as in Remark 1.7, and $\mathfrak{A} = \mathfrak{A}'$.

As in §1, we have an action of the symmetric group $\mathfrak{S}_\beta$, on the set $\mathcal{A}^\beta(w)$, given by $\sigma\mathbf{A} = \big((D,H),\sigma\mathfrak{A}\big)$. We denote by $\sim$ the equivalence relation on $\mathcal{A}^\beta(w)$ generated by $\overset{\iota}{\sim}$, $\overset{\delta}{\sim}$ and the action of $\mathfrak{S}_\beta$ and simply call it equivalence.

Remark 1.33. If $\mathbf{A}\sim\mathbf{A}'$, then $\mathbf{A}$ and $\mathbf{A}'$ have the same number of components and the same weighted genus.

Let $E^\beta_\infty(f)$ denote the set of all excellent mappings defined on some surface, $S \supset \coprod_{h=1}^k \mathbf{S}^1$, such that $\Sigma_F = \coprod_{h=1}^k \mathbf{S}^1$, $F|_{\coprod_{h=1}^k \mathbf{S}^1} = f$ and $\#F^{-1}(\infty) = \beta$. We shall prove the following theorem:

Theorem 1.34. *Let w be the word for f defined by a system of segments. The set $E^\beta_\infty(f)/_{\overset{\mathrm{r}}{\sim}}$ is in one to one correspondence with the set $\mathcal{A}^\beta_m(w)/\sim$.*

Before proving the theorem, we give some elementary remarks.

Lemma 1.35. *Let $\mathbf{A},\mathbf{A}'\in\mathcal{A}^\beta(w)$, $\sigma\in\mathfrak{S}_\beta$. If $\mathbf{A}\overset{\iota}{\sim}\mathbf{A}'$, then $\sigma\mathbf{A}\overset{\iota}{\sim}\sigma\mathbf{A}'$. If $\mathbf{A}\overset{\delta}{\sim}\mathbf{A}'$, then $\sigma\mathbf{A}\overset{\delta}{\sim}\sigma\mathbf{A}'$.*

Remark 1.36. Denote by $\overset{\iota\delta}{\approx}$ the equivalence relation generated by $\overset{\iota}{\sim}$ and $\overset{\delta}{\sim}$. We can consider the quotient relation over $\mathcal{A}^\beta(w)/_{\mathfrak{S}_\beta}$; we denote it by $\overset{\iota\delta}{\approx}$ again. Denote by $[\mathbf{A}]$ the class of $\mathbf{A}$ in $\mathcal{A}^\beta(w)/_{\mathfrak{S}_\beta}$; by Lemma 1.35 we have that $[\mathbf{A}]\overset{\iota\delta}{\approx}[\mathbf{A}'] \iff \exists\,\sigma\in\mathfrak{S}_\beta : \sigma\mathbf{A}\overset{\iota\delta}{\approx}\mathbf{A}'$; furthermore, $\mathcal{A}^\beta_m(w)/\sim \;= \big(\mathcal{A}^\beta_m(w)/_{\mathfrak{S}_\beta}\big)\big/_{\overset{\iota\delta}{\approx}}$.

Remark 1.37. Let (D,H) be a deformation pair and $\widetilde{w}_{(D,H)}$ be the word for $f_{(D,H)}$ defined by the same segments as w. Let $\mathbf{A} = \big((D,H),\mathfrak{A}\big)\in\mathcal{A}^\beta(w)$ and let $\widetilde{\mathfrak{A}}_\mathbf{A}\in\mathcal{A}^\beta(\widetilde{w}_{(D,H)})$ the corresponding assemblage (see Proposition 1.28). By the very definitions, we have $\sigma\widetilde{\mathfrak{A}}_\mathbf{A} = \widetilde{\mathfrak{A}}_{\sigma\mathbf{A}}$ for all σ and $\mathbf{A}$. It follows that the mapping $\widetilde{\mathfrak{A}}$ defines a map $[\widetilde{\mathfrak{A}}] : [\mathbf{A}] \mapsto [\widetilde{\mathfrak{A}}]_{[\mathbf{A}]} = [\widetilde{\mathfrak{A}}_\mathbf{A}]$, which is a bijection between the set of β-assemblages of w having (D,H) as deformation pair, modulo the action of $\mathfrak{S}_\beta$ and the set $\mathcal{A}^\beta(\widetilde{w}_{(D,H)})/_{\mathfrak{S}_\beta}$.

Proof of Theorem 1.34. We begin trying to define a map $\mathbf{A}^\beta_1 : E^\beta_\infty(f) \to \mathcal{A}^\beta_m(w)/_{\mathfrak{S}_\beta}$. Fix $F\in E^\beta_\infty(f)$ and choose a tubular neighborhood, say U, of Σ_F, in such a way that $F|_{\partial(S-U)} = f_{(D,H)}$, for some choice of (D,H). Clearly $F|_{S-U}$ is an immersion that extends $f_{(D,H)}$ and that takes exactly β times the value ∞. Let $\mathfrak{A}^\beta(F)\in\mathcal{A}^\beta_m(\widetilde{w}_{(D,H)})/_{\mathfrak{S}_\beta}$ be the class of minimal assemblages associated to $F|_{S-U}$ by Theorem 0.18. Put $\mathbf{A}^\beta_1(F)\in\mathcal{A}^\beta_m(w)/_{\mathfrak{S}_\beta}$ to be the unique class such that $[\widetilde{\mathfrak{A}}]_{\mathbf{A}^\beta_1(F)} = \mathfrak{A}^\beta(F)$ (Remark 1.37). Observe that $\mathbf{A}^\beta_1(F)$ is not uniquely determined, indeed its construction depends upon:

(a) the choice of the tubular neighborhood;

(b) the choice of the deformation set;

if f_h is a component with no cusps and U_h is the correspondent component of U,

(c) the choice of the branch of ∂U, used as domain of $f^*_{D,1}$.

As we shall see soon, different choices lead to $\overset{\iota\delta}{\sim}$-equivalent classes, thus defining a map $\mathbf{A}^\beta_2 : E^\beta_\infty(f) \to \mathcal{A}^\beta_m(w)/\sim$. To see this, we describe more explicitly the above construction.

Let $\{p_1, \dots, p_\beta\} = F^{-1}(\infty)$ and let $D_1, \dots, D_\beta \subset S$ be disks such that $p_i \in D_i$ and $F(\partial D_i) = g_i(\mathbf{S}_1)$, being $g_1, \dots, g_\beta$ as in Definition 0.10. Denote by Σ^β the set $\Sigma_F \cup \bigcup_{i=1}^\beta \partial D_i$. Since the segments of R are in general position with respect to Γ (condition (4) of Definition 1.18), we see that $\left(F|_{(S-\bigcup \mathring{D}_i)}\right)^{-1}(R)$ is a set of smooth arcs with at least one ending point on Σ^β. Give to all this points the same label as the corresponding point on $\Gamma^\beta = F(\Sigma^\beta)$.

Let γ be an arc of $\left(F|_{(S-\bigcup \mathring{D}_i)}\right)^{-1}(R)$. Clearly, γ intersects ∂U in as many points as $\gamma \cap \Sigma_F$. Label these points as the corresponding ending point of γ. Reading letters on ∂U, we get the word $w_{(D,H)}$, while reading letters on ∂D_i we get the words u_i (for all $i = 1, \dots, k$) of the β expansion (see §1). Pairing together letters corresponding to ending points of the same arc, we get a β-assemblage for $w_{(D,H)}$, hence a β-assemblage for w. By the way we associated an assemblage to an immersion (see the sketch of proof of Theorem 0.18), the just described two constructions lead to the same class in $\mathcal{A}^\beta_m(w)/_{\mathfrak{S}_\beta}$.

From the last construction, it is clear that, leaving U and the deformation set D fixed, but exchanging the role of the two branches of ∂U_h, the assemblage changes by the action of the involution ι_h. It is also clear that choosing a different deformation set, the assemblage changes by a δ-equivalence. Finally if U, U' are two different tubular neighborhood, we may define a map from the set of letters in $w_{(D,H)}$ into itself, simply mapping each letter represented by an intersection of the arc γ with ∂U, to the letter represented by the corresponding intersection of γ with $\partial U'$. It is clear that the assemblages, defined by the two choices, differs up to the action of this map and that this map is a composition of ι-involutions, hence the two assemblages are ι-equivalent.

By the last construction one can easily prove the following:

Proposition 1.38. *If $F, F' \in E^\beta_\infty(f)$ are right equivalent extensions of f, then $\mathbf{A}^\beta_2(F) = \mathbf{A}^\beta_2(F')$.*

By this proposition, $\mathbf{A}^\beta_2$ defines a map: $\mathbf{A}^\beta : E^\beta_\infty(f)/_{\underset{\sim}{\mathrm{r}}} \to \mathcal{A}^\beta_m(w)/\sim$. To end the proof of Theorem 1.34, we prove the following:

Proposition 1.39. *The map $\mathbf{A}^\beta$, is a bijection.*

Proof. $\mathbf{A}^\beta$ is injective. Let $F : S \to \mathbf{N}$ and $F' : S' \to \mathbf{N}$ be such that $\mathbf{A}^\beta(F) = \mathbf{A}^\beta(F')$, then they induce the same deformation pair. By Proposition 1.11 there exist neighborhoods U and U' of $\coprod_{h=1}^k \mathbf{S}^1$ respectively in S and S' and a diffeomorphism $\varphi_1 : U' \to U$ such that $F'|_{U'} = (F|_U) \circ \varphi_1$. On the other hand, $F|_{(S-U)}$ and $F'|_{(S'-U')}$ are two extension of $f_{(D,H)}$ inducing the same class of assemblages in $\mathcal{A}^\beta_m(\widetilde{w}_{(D,H)})/_{\mathfrak{S}_\beta}$, hence, by Theorem 0.18, there exist a diffeomorphism $\varphi_2 : S' - U' \to S - U$ such that $F'|_{(S'-U')} = F|_{(S-U)} \circ \varphi_2$. A local analysis shows that the two diffeomorphisms φ_1 and φ_2 paste together, defining a diffeomorphism $\varphi : S' \to S$ such that $F' = F \circ \varphi$.

To prove that $\mathbf{A}^\beta$ is surjective we need the following lemma:

LEMMA 1.40. *If w has minimal assemblages, then all cusps of f are negative.*

PROOF. By contradiction, suppose that f has a positive cusp, then every deformation of f has a positive curl. Hence, by Lemma 0.22, $\mathcal{A}^\beta_m(\widetilde{w}_{(D,H)}) = \varnothing$ for all deformation pair (D, H). By Remark 1.31, this means that $\mathcal{A}^\beta_m(w) = \varnothing$. $\square$

Finally we prove that $\mathbf{A}^\beta$ is surjective. Fix $\mathbf{A} = \big((D,H), \mathfrak{A}\big) \in \mathcal{A}^\beta_m(w)$, by the previous lemma, f has a coherent side, hence, by Proposition 1.11, there exists a local, excellent extension $F_1 : M \to \mathbf{N}$ of f such that $F_1|_{\partial M} = f_{(D,H)}$. By Proposition 1.28, $\widetilde{\mathfrak{A}}_{\mathbf{A}}$ is a minimal β-assemblage of $\widetilde{w}_{(D,H)}$, hence, by Theorem 0.18, there exist a surface $\widetilde{S}$ and an immersion $F_2 : \widetilde{S} \to \mathbf{N}$ such that $\partial\widetilde{S} = \coprod_{h=1}^{k_H} \mathbf{S}^1$ and $F_2|_{\partial\widetilde{S}} = f_{(D,H)}$. Once again a local analysis shows that M and $\widetilde{S}$ can be pasted along their boundary in such a way that the resulting map F, obtained by pasting F_1 and F_2, is an excellent map extending f. By construction, $\mathbf{A}^\beta(F) = \mathbf{A}$. $\square$

2. Factorization of excellent mappings

Classically the problem to find out a factorization $F = \pi \circ F_1$ for an excellent map $F : S \to \mathbf{N}$ into a given line bundle $\pi : \mathbf{E} \to \mathbf{N}$ was posed asking for F_1 to be an immersion. This problem was solved first by Haefliger [11] in the case $\mathbf{N} = \mathbb{R}^2$ and $\mathbf{E} = \mathbb{R}^3$; successively Millet [15] remarked that Hafliger's proof worked for an arbitrary surface $\mathbf{N}$ and the trivial bundle ($\mathbf{E} = \mathbf{N} \times \mathbb{R}$) too. Finally, the author [12] used again Hafliger's methods to solve the general case: let C be a component of Σ_F, and put $c_C = \#\{\text{cusps in } C\}$, $\nu_C = \pm 1$ according as C has a trivial normal bundle or not and $\varepsilon_C = \pm 1$ according as $F|_C{}^*\mathbf{E} \to C$ is the trivial bundle or not. The following theorem holds:

THEOREM. *F factors through an immersion into $\mathbf{E}$ if and only if for all components C of Σ_F, $(-1)^{c_C}\nu_C\varepsilon_C = 1$.*

As announced in the introduction, in this section we deal with the problem of finding a factorization, $F = \pi \circ F_1$, asking for F_1 to be an embedding. We begin with some lemmas and propositions, that we shall use later.

From now on Σ and Γ will denote the set of critical points and the apparent contour of an excellent map.

LEMMA 2.1. *Let $F : S \to \mathbf{N}$ be an excellent map and let $D \subset \mathbf{N}$ be an embedded disk such that: ∂D is in general position with respect to Γ, D contains no cusp and $D \cap \Gamma$ is a union of simple arcs. Then F restricted to any component of $F^{-1}(D) - \Sigma$ is injective.*

PROOF. Fix a point $\infty \notin D$ and a vector field X on $\mathbf{N}$, vanishing at most at ∞. Let C be a connected component of $F^{-1}(D) - \Sigma$ and let C_ε be obtained by removing a small neighborhood of Σ. The following facts hold: $F|_{C_\varepsilon}$ is an immersion; ∂C_ε is made up by branches of $F^{-1}(\partial D)$ and arcs parallel to Σ; if k is the number of boundary components of ∂C_ε, then $F(\partial C_\varepsilon)$ is made up by k loops obtained by arcs of ∂D and branches of Γ. It is easy to see that each of these loops, endowed with the side induced by the immersion, has rotation number greater or equal to 1, hence $R_X\big(F|_{\partial C_\varepsilon}\big) \geq k$. By (0-1), $R_X\big(F|_{\partial C_\varepsilon}\big) = \chi(C_\varepsilon) \leq 1$, hence $k = 1$, C_ε is a disk and $F|_{\partial C_\varepsilon}$ cannot have any crossing. Thus $F|_{C_\varepsilon}$ is 1 to 1. An exhaustion argument ends the proof. $\square$

Let R be a system of segments for $F|_\Sigma$. The set $F^{-1}(R)$ is made up by smooth, oriented arcs having at least the ending point on $\Sigma \cup F^{-1}(\infty)$. Denote by $H(F)$ the set of all such arcs starting from $\Sigma \cup F^{-1}(\infty)$.

Remark 2.2. Since $\omega \in \Omega$ is a closed loop based at ∞, $F^{-1}(\Omega) \subseteq H(F)$.

Lemma 2.3. *F restricted to each component of $S - H(F)$ is injective.*

Proof. By contradiction, suppose $p, q \in S - H(F)$ be such that $F(p) = F(q)$ and let $\gamma : [0,1] \to S - H(F)$ be a path joining p and q. It is not hard to see that we can suppose that $p, q \in S - F^{-1}(R)$, $\gamma([0,1]) \subset S - F^{-1}(R)$ and $F \circ \gamma$ is a simple curve in general position with respect to Γ (see [10]). Let $D \subset \mathbf{N}$ be the disk bounded by $F \circ \gamma$. Since $\partial D \cap R = \varnothing$, $D \cap R = \varnothing$, and since R is a system of segments for $F|_\Sigma$, D contains neither cusp nor curls of Γ. A contradiction follows by Lemma 2.1. $\square$

We now state two topological lemmas, whose proof is an easy exercise.

Lemma 2.4. *Let X be an arcwise connected topological space, $U, V \subset X$ two open subset such that $X = U \cup V$ and $U \cap V$ is not arcwise connected. Then $H_1(X) \neq 0$, where H_1 denotes the first homology group.*

Lemma 2.5. *Let $U_1, \ldots, U_n$, $n \geq 3$, be open subsets of a topological space X. If $U_i \cap U_j \neq \varnothing \iff |i - j| \equiv 0, 1 \mod n$. Then either $\bigcap_{i=1}^n U_i \neq \varnothing$, and in this case $n = 3$, or $U_i \cap \big(\bigcup_{i \neq j} U_j\big)$ is disconnected $\forall i$.*

From the above two lemmas, we have the following:

Proposition 2.6. *Let $U_1, \ldots, U_n$ ($n \geq 3$) be arcwise connected, open subsets of the topological space X, fulfilling the hypothesis of Lemma 2.5. Then either $n = 3$ and $\bigcap_i U_i \neq \varnothing$ or $H_1\big(\bigcup_i U_i\big) \neq 0$.*

From now on, $\mathcal{C}$ will denote the set of connected components of $S - H(F)$.

Proposition 2.7. *Every $C \in \mathcal{C}$ is diffeomorphic to an open disk.*

Proof. Let $C \in \mathcal{C}$. As already remarked, $F^{-1}(\Omega) \subseteq H(F)$, hence $F(C) \subset \mathbf{N} - \Omega$. Since Ω is a minimal system of generators of $\pi_1(\mathbf{N}, \infty)$, $\mathbf{N} - \Omega$ is diffeomorphic to an open disk, hence, by Lemma 2.3, we can say that C is a planar surface, that is an open disk with some holes. By contradiction, assume $F(C)$ is not an open disk, hence $F(C)$ has at least two boundary components, furthermore the boundary of $F(C)$ is made up by branches of Γ and branches of segments of R. Let γ be one of the internal components of $\partial F(C)$. If γ contains a branch of the segment r, since the segment ends at ∞, the segment itself cuts $F(C)$ from boundary to boundary. But this contradicts the fact that C does not contain arcs in $F^{-1}(R)$ starting and ending at its boundary. If γ is made up only by branches of Γ, then it bounds a connected component of $\mathbf{N} - \Gamma$. The segment starting from this connected component, leads to a contradiction as in the previous case. $\square$

PROPOSITION 2.8. *Let C_1, $C_2 \in \mathcal{C}$, then $F(C_1) \cap F(C_2)$ is connected.*

PROOF. By contradiction, suppose $F(C_1) \cap F(C_2)$ is disconnected. By Lemma 2.4, $F(C_1) \cup F(C_2)$ has at least two boundary components. A contradiction follows as in the proof of the previous proposition. $\square$

Factorization. Let S and $\mathbf{N}$ be surfaces and $\pi : \mathbf{E} \to \mathbf{N}$ a line bundle.

DEFINITION 2.9. We will say that a map $F : S \to \mathbf{N}$ factors through π, if there exists an embedding $F_1 : S \to \mathbf{E}$ such that $F = \pi \circ F_1$.

REMARK 2.10. Finding a map F_1 such that $F = \pi \circ F_1$ is the same as finding a section σ of the line bundle $F^*\mathbf{E}$, induced by F (see the diagram below). Hence F factors through π if and only if there exists a section $\sigma : S \to F^*\mathbf{E}$ of the induced bundle, such that $\pi^*F \circ \sigma$ is an embedding. If $\mathbf{E} = \mathbf{N} \times \mathbb{R}$ is the trivial bundle, this condition reduce to the existence of a function $h : S \to \mathbb{R}$ such that $(F, h) : S \to \mathbf{N} \times \mathbb{R}$ is an embedding. We shall call such a function an height for F.

$$\begin{tikzcd}
F^*\mathbf{E} \arrow[r, "\pi^*F"] \arrow[d, shift left, "F^*\pi"] & \mathbf{E} \arrow[d, "\pi"] \\
S \arrow[u, shift left, "\sigma"] \arrow[ur, "F_1"] \arrow[r, "F"'] & \mathbf{N}
\end{tikzcd}$$

REMARK 2.11. Suppose $F : S \to \mathbf{N}$ and $F' : S' \to \mathbf{N}$ are right equivalent maps, it is easy to see that F factors through π if and only if F' do. For excellent maps this means that conditions in orders that a map factors through π "must" be contained in the assemblage associated to the map itself.

Let $F^{-1}(\infty) = \{p_1, \ldots, p_\beta\}$ and let $D_1, \ldots, D_\beta$ disks in S such that $p_i \in D_i$ and $F|_{\partial D_i} = g_i$ (g_i as in Definition 0.10), and let $\widetilde{S} = S - \bigcup_{i=1}^{\beta} \mathring{D}_i$, $\widetilde{F} = F|_{\widetilde{S}}$.

PROPOSITION 2.12. *F factors through π, if and only if $\widetilde{F}$ do.*

PROOF. Let $\widetilde{F}_1 : \widetilde{S} \to \mathbf{E}$ be an embedding such that $\pi \circ \widetilde{F}_1 = \widetilde{F}$ and let U_∞ be an open ball around ∞, such that $U_\infty \cap \Gamma = \varnothing$ and containing all curves g_i. Let U_i the connected component of $F^{-1}(U_\infty)$ containing p_i. Clearly, $U_i \cap U_j = \varnothing$ for all $i \neq j$, and F is a diffeomorphism between U_i and U_∞. Take a trivialization $\Phi : \mathbf{E}|_{U_\infty} \to U_\infty \times \mathbb{R}$, then $\Phi \circ \widetilde{F}_1\big|_{\bigcup_i (U_i - \mathring{D}_i)}(p) = \big(F(p), h(p)\big)$. Since $\widetilde{F}_1$ is injective, we may suppose that h assumes constant values, say c_i, on $U_i - D_i$, such that $c_i \neq c_j$ for all $i \neq j$. Extend $\widetilde{F}_1$, to the required embedding F_1, defining $F_1(p) = \Phi^{-1}\big(F(p), c_i\big)$ for all $p \in D_i$. $\square$

The trivial bundle. First of all we consider the case $\mathbf{E} = \mathbf{N} \times \mathbb{R}$. In Remark 2.10 we saw that F factors through $\mathbf{E} = \mathbf{N} \times \mathbb{R}$ if and only if there exists an height function h for F.

PROPOSITION 2.13. *Let h be an height function for F, then for all C_1, $C_2 \in \mathcal{C}$ such that $F(C_1) \cap F(C_2) \neq \varnothing$, one (and only one) of the following two holds:*

(1) $\forall x \in C_1,\ y \in C_2 \quad F(x) = F(y) \implies h(x) < h(y)$;
(2) $\forall x \in C_1,\ y \in C_2 \quad F(x) = F(y) \implies h(x) > h(y)$.

PROOF. Clearly at most one of the two holds. By contradiction, suppose that both are false, then there exist $x_1, x_2 \in C_1$, $y_1, y_2 \in C_2$ such that $F(x_1) = F(y_1) = z_1$, $F(x_2) = F(y_2) = z_2$ and $h(x_1) < h(y_1)$, $h(x_2) > h(y_2)$. By Proposition 2.8 $F(C_1) \cap F(C_2)$ is connected, let $\gamma : [0,1] \to F(C_1) \cap F(C_2)$ be a path joining z_1 and z_2, and let γ_1 and γ_2 its liftings to C_1 and C_2 respectively (such liftings exists by Lemma 2.3). Put $h_i = h \circ \gamma_i$, then $h_1(0) = h(x_1) < h(y_1) = h_2(0)$ and $h_1(1) = h(x_2) > h(y_2) = h_2(1)$, therefore there exists $t_0 \in (0,1)$ such that $h_1(t_0) = h_2(t_0)$. Call $p_1 = \gamma_1(t_0) \in C_1$, $p_2 = \gamma_2(t_0) \in C_2$, then $p_1 \neq p_2$ and $F(p_1) = F(p_2)$, $h(p_1) = h(p_2)$, that is a contradiction. $\square$

If h is an height for F we can give a structure of oriented graph to the set $\mathcal{C}$: C_1, $C_2 \in \mathcal{C}$ are joined by an edge pointing to C_2 (in symbols $C_1 \to C_2$) if and only if C_1 and C_2 verify (1) in Proposition 2.13. We denote by $L(h)$ this oriented graph.

PROPOSITION 2.14. *The graph $L(h)$ has no loops.*

PROOF. By contradiction, let $C_1 \to C_2 \to \cdots \to C_n \to C_1$ be a loop of minimal length. Observe that $F(C_i) \cap F(C_j) \neq \varnothing \iff |i-j| \equiv 0, 1 \mod n$, on the contrary we could use Proposition 2.13 to find a shorter loop. By Proposition 2.6, either $n = 3$ and $\bigcap_i F(C_i) \neq \varnothing$ or $H_1(\bigcup_i F(C_i)) \neq 0$. In the first case, take points $x_i \in C_i$ such that $F(x_1) = F(x_2) = F(x_3)$, by definition of $\to$, $h(x_1) < h(x_2)$, $h(x_2) < h(x_3)$ and $h(x_3) < h(x_1)$, that is a contradiction. In the second case, a contradiction follows as in the proof of Proposition 2.7. $\square$

As in the definition of the word of a curve, for all $r \in R$, fix an orientation of a small tubular neighborhood of $r - \infty$.

PROPOSITION 2.15. *Let C_1, C_2, C_3, $C_4 \in \mathcal{C}$ and ℓ, ℓ_1, ℓ_2, be edges of the graph $H(F)$. The following two holds:*

(A) *suppose that C_1, C_2 paste along ℓ (i.e. $\bar{C}_1 \cap \bar{C}_2 \supseteq \ell$) and $F(C_3) \cap F(\ell) \neq \varnothing$. Then either $C_1 \to C_3$ and $C_2 \to C_3$ or $C_3 \to C_1$ and $C_3 \to C_2$;*
(B) *suppose that C_1, C_2 paste along ℓ_1, C_3, C_4 paste along ℓ_2 and $F(\ell_1) \cap F(\ell_2) \neq \varnothing$. Let $r \in R$ be the segment containing $F(\ell_i)$ and suppose that $F(C_1)$, $F(C_3)$ stand on the left side of r while $F(C_2)$, $F(C_4)$ stand on its right side. Then either $C_1 \to C_3$ and $C_2 \to C_4$ or $C_3 \to C_1$ and $C_4 \to C_2$.*

PROOF. Let C_1, C_2, C_3 be as in (A), then $F(C_1) \cap F(C_3)$ and $F(C_2) \cap F(C_3)$ are both not empty, hence some edge of $L(h)$ joins C_1, C_3 and C_2, C_3. By contradiction, suppose (A) is false, say $C_1 \to C_3$ and $C_3 \to C_2$. Let $z \in F(C_3) \cap F(\ell)$ and let $p \in \ell$ such that $F(p) = z$. Take a small neighborhood U of p, such that $U \cap C_3 = \varnothing$ and $F(U) \subset F(C_3)$, and a path γ in U joining two points $p_1 \in C_1$, $p_2 \in C_2$. Let γ_3 be the lifting of $F \circ \gamma$ to C_3. Clearly $F \circ \gamma_3 = F \circ \gamma$, but $C_1 \to C_3 \implies h(\gamma(0)) < h(\gamma_3(0))$ and $C_3 \to C_2 \implies h(\gamma(1)) > h(\gamma_3(1))$. As in the previous proposition this fact leads to a contradiction. A completely analogous argument proves (B). $\square$

Proposition 2.14 and Proposition 2.15 give necessary conditions in order that an excellent map F may factor through $\mathbf{N}\times\mathbb{R}$. We now prove that these conditions are sufficient too.

THEOREM 2.16. *Let $F : S \to \mathbf{N}$ be an excellent map. F factors through $\mathbf{N}\times\mathbb{R}$ if and only if the set $\boldsymbol{\mathcal{C}}$ can be given a structure of oriented graph with no loops, fulfilling conditions* (A) *and* (B) *of Proposition 2.15.*

PROOF. Let $\widetilde{S}$ be as in Proposition 2.12. We proceed in two steps: *Step 1:* definition of an height on $\widetilde{S}$ minus a tubular neighborhood of Σ; *Step 2:* extension of such an height to the tube. The thesis will follow by Proposition 2.12.

Let L the graph in the assumption, since it has no loops, $\to$ extends to an ordering on $\boldsymbol{\mathcal{C}}$, call $\prec$ such an ordering. Clearly $\prec$ fulfills (A) and (B). Put $\boldsymbol{\mathcal{C}} = \{C_1, \ldots, C_n\}$, being $C_s \prec C_{s+1}$ for all s.

Step 1. Let U be a tubular neighborhood of Σ such that $F|_{\partial U}$ is a generic curve and denote $S' = \widetilde{S} - U$, $\Gamma' = F(\partial S')$ and if $C \in \boldsymbol{\mathcal{C}}$ denote $C' = C \cap S'$. Let V_∞ be a ball around ∞, contained in the internal side of all curves g_i (see Definition 0.10), that is to say $F(\widetilde{S}) \subset (\mathbf{N} - V_\infty)$. For all $r \in R$ fix an oriented neighborhood V_r of $r - V_\infty$ in $\mathbf{N} - V_\infty$ and a diffeomorphism $\varphi_r : V_r \to [0, t_r] \times [-1, 1]$, such that:

(1) $V_r \cap V_{r'} = \varnothing \quad \forall r \neq r'$;
(2) $\forall\, r \in R$, V_r contains no crossings of Γ';
(3) $\forall\, r \in R$, $\varphi_r(r) \subseteq [0, t_r] \times \{0\}$;
(4) $\forall\, r \in R$, $(r \cup \Gamma') \cap V_r$ is connected (this fact and (2) say that V_r contains only branches of Γ' intersecting r);
(5) let $\Gamma_{r,j}$ be the component of $\Gamma' \cap V_r$ intersecting r at the j^{th} point, counting according the orientation of r, then $\varphi_r(\Gamma_{r,j}) = \{j\} \times [-1, 1]$;
(6) $\forall\, r \in R$, φ_r and $\varphi_{r|r} : r \to \mathbb{R}$ are orientation preserving.

Put $V = \bigcup_{r\in R} V_r$, for any proper edge ℓ of $H(F)$, denote by U_ℓ the connected component of $F^{-1}(V) \cap S'$ containing $\ell \cap S'$ and by $r(\ell)$ the segment containing $F(\ell)$, let $s(\ell)$ and $t(\ell)$ be the two indexes such that $C_{s(\ell)}$, $C_{t(\ell)} \in \boldsymbol{\mathcal{C}}$ are the two components pasting along ℓ, the former on the left of ℓ, the latter on the right (left and right with respect to the orientations induced by F). For all $C \in \boldsymbol{\mathcal{C}}$ denote $C^* = (C \cap S') - \bigcup_\ell U_\ell$. Finally let $\psi : \mathbb{R} \times [-1, 1] \to \mathbb{R}$ be a C^∞ function such that: $\psi(x, y) = 0\ \forall y \in [-1, -2/3]$, $\psi(x, y) = 1\ \forall y \in [2/3, 1]$ and $\psi(x, y) \in [0, 1]\ \forall x, y$. We define the function $h_1 : S' \to \mathbb{R}$,

$$h_1(x) = \begin{cases} s & \text{if } x \in C_s^* \\ s(\ell) + \big(t(\ell) - s(\ell)\big)\psi\Big(\varphi_{r(\ell)}\big(F(x)\big)\Big) & \text{if } x \in U_\ell. \end{cases} \tag{2-1}$$

Clearly such a function is smooth. We now prove that $(F|_{S'}, h_1) : S' \to \mathbf{N} \times \mathbb{R}$ is injective. Let $x \neq y \in S'$, three cases appear:

(I) $x \in C_s^*$ and $y \in C_{s'}^*$; **(II)** $x \in C_s^*$ and $y \in U_\ell$; **(III)** $x \in U_\ell$ and $y \in U_{\ell'}$.

Case (I). If $s = s'$, by Lemma 2.3, $F(x) \neq F(y)$; if $s \neq s'$, clearly $h_1(x) \neq h_1(y)$.
Case (II). If $F(x) \neq F(y)$ there is nothing to prove. If $F(x) = F(y)$, then $F(C_s^*) \cap F(U_\ell) \neq \varnothing$, therefore, by the way we chose the V_r's, $F(C_s) \cap r(\ell) \neq \varnothing$; by (A), this means either $C_s \prec C_{s(\ell)}$ and $C_s \prec C_{t(\ell)}$ or $C_{s(\ell)} \prec C_s$ and $C_{t(\ell)} \prec C_s$,

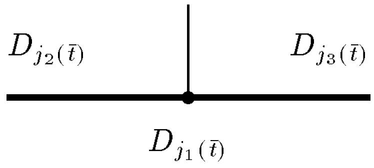

FIGURE 12

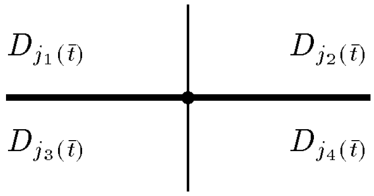

FIGURE 13

that is to say either $s < s(\ell)$ and $s < t(\ell)$ or $s(\ell) < s$ and $t(\ell) < s$. By definition of h_1, $\min\{s(\ell), t(\ell)\} < h_1(y) < \max\{s(\ell), t(\ell)\}$, hence $s = h_1(x) \neq h_1(y)$.

Case (III). Suppose $F(x) = F(y) = p$. The V_r's are pairwise disjoint, hence $r(\ell) = r(\ell')$ and since $F(U_\ell) \cap F(U_{\ell'}) \neq \varnothing$, $F(\ell) \cap F(\ell') \neq \varnothing$. By (B), either $C_{s(\ell)} \prec C_{s(\ell')}$ and $C_{t(\ell)} \prec C_{t(\ell')}$ or $C_{s(\ell')} \prec C_{s(\ell)}$ and $C_{t(\ell')} \prec C_{t(\ell)}$, that is to say either $s(\ell) < s(\ell')$ and $t(\ell) < t(\ell')$ or $s(\ell') < s(\ell)$ and $t(\ell') < t(\ell)$. Using the definition of h_1, an easy computation shows that either $h_1(x) < h_2(y)$ or $h_2(y) < h_1(x)$. This ends the first step.

Step 2. For the sake of simplicity, we suppose Σ is connected, it will be clear that the same proof works in the general case too. Let U be as above and put

$$M = \begin{cases} \mathbb{R} \times [-1,1]/_{(x,y)=(x+1,y)} & \text{if } U \text{ is orientable,} \\ \mathbb{R} \times [-1,1]/_{(x,y)=(x+1,-y)} & \text{if } U \text{ is unorientable.} \end{cases}$$

Let $\varphi : M \to U$ be a diffeomorphism such that $\varphi(\{y = 0\}) = \Sigma$ and the inverse image of $U \cap H(F)$ is made up by vertical segments, that is if $p : \mathbb{R} \times [-1,1] \to M$ denotes the quotient map, we suppose that $(\varphi \circ p)^{-1}(H(F)) \cap [0,1] \times [-1,1]$ has the form: $\bigcup_i (\{t_i\} \times [0,1]) \cup \bigcup_j (\{s_j\} \times [-1,0]) \cup \bigcup_l (\{z_l\} \times [-1,1])$. Denote by D_j both the generic component of $M - \varphi^{-1}(H(F))$ and the corresponding component of $U - H(F)$ and by $C_{i(j)} \in \mathcal{C}$ the component of $S - H(F)$ containing D_j. The following two lemmas are an immediate consequence of (A) and (B) respectively.

LEMMA 2.17. *Let $\bar t \in [0.1)$ a cusp point (with the just said notations, either $\bar t = t_i$ or $t = s_j$) and let $D_{j_1(\bar t)}$, $D_{j_2(\bar t)}$, $D_{j_3(\bar t)}$ the three components having $(\bar t, 0)$ in their closure. Let $D_{j_1(\bar t)}$ the one opposed to the edge starting at $(\bar t, 0)$ (see Figure 12); then either $C_{i(j_2(\bar t))} \prec C_{i(j_1(\bar t))} \prec C_{i(j_3(\bar t))}$ or $C_{i(j_3(\bar t))} \prec C_{i(j_1(\bar t))} \prec C_{i(j_2(\bar t))}$.*

LEMMA 2.18. *Let $D_{j_1(z_l)}, D_{j_2(z_l)}, D_{j_3(z_l)}, D_{j_4(z_l)}$ be the four components having $(z_l, 0)$ in their closure (numbered as in Figure 13); then either $C_{i(j_1(z_l))} \prec C_{i(j_3(z_l))}$ and $C_{i(j_2(z_l))} \prec C_{i(j_4(z_l))}$ or $C_{i(j_1(z_l))} \succ C_{i(j_3(z_l))}$ and $C_{i(j_2(z_l))} \succ C_{i(j_4(z_l))}$.*

Denote $h_{1,-1}(t) = h_1 \circ p(t, -1)$, $h_{1,1}(t) = h_1 \circ p(t, 1)$. By the previous two lemmas, and the way we constructed h_1 in Step 1, we can suppose $h_{1,1}$ and $h_{1,-1}$ have the following properties:

(1) $h_{1,1}(t+1) = h_{1,1}(t)$ and $h_{1,-1}(t+1) = h_{1,-1}(t)$ if U is orientable, otherwise $h_{1,1}(t+1) = h_{1,-1}(t)$ and $h_{1,-1}(t+1) = h_{1,1}(t)$;
(2) there exists $\varepsilon > 0$ such that:
 a) $h_{1,1}$ is strictly monotonic on $[t_i - \varepsilon, t_i + \varepsilon]$ and $[z_l - \varepsilon, z_l + \varepsilon]$;
 b) $h_{1,-1}$ is strictly monotonic on $[s_j - \varepsilon, s_j + \varepsilon]$ and $[z_l - \varepsilon, z_l + \varepsilon]$;
 c) $h_{1,1}$ e $h_{1,-1}$ assume constant values on the other intervals;
(3) $\forall\, t \in [0,1) \quad h_{1,1}(t) = h_{1,-1}(t) \iff$ either $t = t_i$ or $t = s_j$ for some i or j.

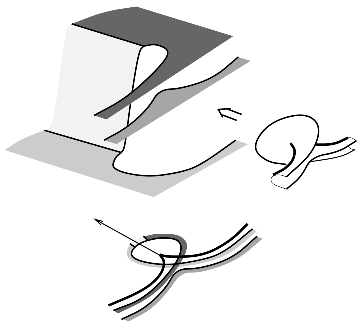

FIGURE 14

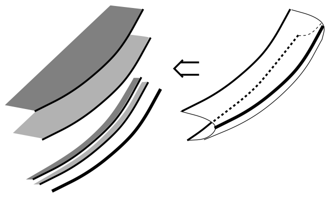

FIGURE 15

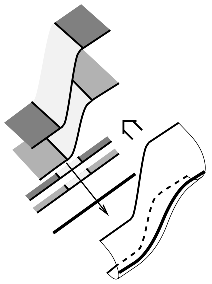

FIGURE 16

These conditions imply that we can extend h_1 in a neighborhood $[\bar{t}-\delta, \bar{t}+\delta]\times[-1,1]$ of each cusp point $(\bar{t}, 0)$, as suggested in Figure 14, in such a way that $(F, h_1) : [\bar{t}-\delta, \bar{t}+\delta]\times[-1,1] \to \mathbf{N}\times\mathbb{R}$ is an embedding. Moreover, if $D_{j_l(\bar{t})}$ are the three components having $(\bar{t}, 0)$ in their closure and $M(\bar{t}) = \max\{i(j_2(\bar{t})), i(j_3(\bar{t}))\}$, $m(\bar{t}) = \min\{i(j_2(\bar{t})), i(j_3(\bar{t}))\}$, we can suppose that

$$m(\bar{t}) \le h_1(t, y) \le M(\bar{t}) \quad \forall\, (t, y) \in [\bar{t}-\delta, \bar{t}+\delta]\times[-1,1]; \tag{2-2}$$

and, up to choose U and δ small enough, that

$$F\big([\bar{t}_1-\delta, \bar{t}_1+\delta]\times[-1,1]\big) \cap F\big([\bar{t}_2-\delta, \bar{t}_2+\delta]\times[-1,1]\big) = \varnothing \tag{2-3}$$

for all pairs of cusp points $(\bar{t}_1, 0)$, $(\bar{t}_2, 0)$.

LEMMA 2.19. *The mapping* $(F, h_1) : S' \cup \bigcup_{\bar{t}}\big([\bar{t}-\delta, \bar{t}+\delta]\times[-1,1]\big) \to \mathbf{N}\times\mathbb{R}$ *is an embedding.*

PROOF. By the way we extended h_1, the map is clearly an immersion. By contradiction, suppose that there exist x_1, $x_2 \in S' \cup \bigcup_{\bar{t}}[\bar{t}-\delta, \bar{t}+\delta]\times[-1,1]$ such that $F(x_1) = F(x_2)$ and $h_1(x_1) = h_1(x_2)$. Since (F, h_1) restricted to both S' and $[\bar{t}-\delta, \bar{t}+\delta]\times[-1,1]$ is injective and by (2-3), we see that $x_1 \in [\bar{t}-\delta, \bar{t}+\delta]\times[-1,1]$ and $x_2 \in S'$. By construction, h_1 assumes a constant value in a neighborhood of x_1, let C_i be the component of $S' - H(F)$ containing it. $F(C_i)$ contains the cusp corresponding to $(\bar{t}, 0)$, hence $F(C_i) \cap F(D_{j_l(\bar{t})}) \neq \varnothing$ for all $l = 1, 2, 3$. Use twice (A) and get either $i < m(\bar{t}) = \min_l i(j_l(\bar{t}))$ or $i > M(\bar{t}) = \max_l i(j_l(\bar{t}))$. Using (2-2) and the fact that $h_1(x_2) = i$, we get a contradiction. □

Only remains to extend h_1 over $U - \bigcup_{\bar{t}}\big([\bar{t}-\delta, \bar{t}+\delta]\times[-1,1]\big)$. If both $h_{1,1}$ and $h_{1,-1}$ assume constant values, an extension can be easily found as suggested in Figure 15. Furthermore, conditions (2) and (3) ensure we can define local extensions of h_1, over the sets $[z_l-\varepsilon, z_l+\varepsilon]\times[-1,1]$, in such a way that the value of such an extension at the point (t, y) is between $h_{1,1}(t)$ and $h_{1,-1}(t)$ (see Figure 16). Clearly all such extensions can be pasted to give a function $h : \widetilde{S} \to \mathbb{R}$. An argument in all similar to the one used to prove the previous lemma, proves the following too:

LEMMA 2.20. *The map* $(F, h) : S \to \mathbf{N}\times\mathbb{R}$ *is injective.*

Since (F, h) is an immersion, this ends the proof of Theorem 2.16. □

Now, we give a description of line bundles over surfaces, fitting in with our purpose to find conditions in order that an excellent mapping may factor through a line bundle.

Line bundles. Let $\mathbf{N}_1$ be a connected, compact surface with one boundary component and $\pi_1 : \mathbf{E}_1 \to \mathbf{N}_1$ a line bundle. Suppose $\mathbf{E}_1|_{\partial\mathbf{N}_1}$ is the trivial bundle and let $\Phi : \mathbf{S}^1 \times \mathbb{R} \to \mathbf{E}_1|_{\partial\mathbf{N}_1}$ a trivialization. Denote by φ the restriction of Φ to the null section. We can consider the line bundle over $\mathbf{N} = \mathbf{N}_1 \cup_\varphi \mathbf{D}^2$, whose total space is $\mathbf{E} = \mathbf{E}_1 \cup_\Phi (\mathbf{D}^2 \times \mathbb{R})$ and whose projection is the obvious one. Here we have denoted by $X \cup_f Y$ the union of X and Y along the mapping $f : A \subset X \to Y$.

Definition 2.21. The bundle constructed this way will be called the closure of $\mathbf{E}_1$ and denoted by $\widehat{\mathbf{E}}_1$.

If $\mathbf{N}$ is a compact, connected surface without boundary and $D \subset \mathbf{N}$ is an embedded disk, then $\mathbf{E}|_D$ is the trivial bundle, hence the following hold:

Proposition 2.22. *Every line bundle over* $\mathbf{N}$ *is the closure of some bundle over a compact, connected surface with one boundary component.*

Since the only line bundle over $\mathbf{S}^2$ is the trivial one, from now on we shall suppose that $g(\mathbf{N}) > 0$. Put $h(\mathbf{N}) = 2g(\mathbf{N})$ if $\mathbf{N}$ is orientable, $h(\mathbf{N}) = g(\mathbf{N})$ otherwise. It is a classical fact in topology, that a surface $\mathbf{N}$ is the quotient of a $2h(\mathbf{N})$-gon, by pairwise identification of its edges, and such a polygon can be found by cutting $\mathbf{N}$ along a minimal system of generators of its fundamental group. Fix the following data: a point $\infty \in \mathbf{N}$, a set $\Omega = \{\omega_i\}$ of smooth curves, as in (2) of Definition 1.18 and a small open ball U_∞ around the point ∞. Put $\mathbf{N}_1 = \mathbf{N} - U_\infty$, and let $P_{\mathbf{N}_1}$ be the $4h$-gon resulting by cutting $\mathbf{N}_1$ along all the curves $\omega_i \cap \mathbf{N}_1$. Call ℓ_i, ℓ_i' the two edges resulting by the cut along ω_i. For all i, let $\psi_i : \ell_i \to \ell_i'$ be the diffeomorphism giving the identification. Suppose to be given a distribution of weights, $\rho_i = \pm 1$, on the set Ω, and define $\mathbf{E}_\rho = P_{\mathbf{N}_1} \times \mathbb{R}/\sim$, being $(x_1, t_1) \sim (x_2, t_2)$ if and only if either the two pairs are equal or $x_1 \in \ell_i, x_2 = \psi_i(x_1)$ and $t_2 = \rho_i t_1$ for some i. $\mathbf{E}_\rho$, endowed with the obvious projection, is a line bundle over $\mathbf{N}_1$.

Proposition 2.23. *The line bundle* $\mathbf{E}_\rho|_{\partial\mathbf{N}_1}$ *is the trivial bundle.*

Proof. Such a bundle is obtained successively pasting $2h$ copies of $[0,1] \times \mathbb{R}$, by means of mappings of the following two kinds: $(1,t) \mapsto (0,t)$, $(1,t) \mapsto (0,-t)$, and both kinds of such identifications are even in number. $\square$

By the previous proposition and Proposition 2.22 we see that every distribution of weights, ρ, over Ω generates a line bundle over $\mathbf{N}$, simply taking the closure, $\widehat{\mathbf{E}}_\rho$, of the just constructed bundle $\mathbf{E}_\rho$. Simple technical arguments prove the following:

Proposition 2.24. *If* $\mathbf{E}$ *is a line bundle over* $\mathbf{N}$*, then there exist a distribution of weights* ρ *over* Ω *such that* $\mathbf{E}$ *is isomorphic to* $\widehat{\mathbf{E}}_\rho$*.*

The general case. We turn back to our problem. Let $F : S \to \mathbf{N}$ be an excellent map and $\pi : \mathbf{E} \to \mathbf{N}$ a line bundle. Let R, U_∞, $\widetilde{S}$ and $\mathbf{N}_1 = \mathbf{N} - U_\infty$ be as above. By Proposition 2.24, we can suppose $\mathbf{E} = \widehat{\mathbf{E}}_\rho$ for some distribution of weights ρ, hence by Proposition 2.12 and Remark 2.10, our problem reduces to find a section σ of $\widetilde{F}^*\mathbf{E}_\rho \to \widetilde{S}$, making $\pi_\rho^*\widetilde{F} \circ \sigma$ into an embedding. First of all we try to understand what "finding a section of the line bundle $\widetilde{F}^*\mathbf{E}_\rho$" means. Let S' be the surface obtained from $\widetilde{S}$ cutting along $\widetilde{F}^{-1}(\Omega)$, and let $\xi : S' \to \widetilde{S}$ be the quotient map given by the cut. Let $\varphi : P_{\mathbf{N}_1} \to \mathbf{N}_1$ be the mapping generated by the cut of $\mathbf{N}_1$

along Ω, then there exists a unique map $F' : S' \to P_{\mathbf{N}_1}$ such that $\varphi \circ F' = \widetilde{F} \circ \xi$. By this fact, by the way $\mathbf{E}_\rho$ was defined and by the following commutative diagram

$$\begin{array}{ccccccccc} P_{\mathbf{N}_1} \times \mathbb{R} & \longleftarrow & \xi^* \widetilde{F}^* \mathbf{E}_\rho & \longrightarrow & \widetilde{F}^* \mathbf{E}_\rho & \longrightarrow & \mathbf{E}_\rho & \longleftarrow & P_{\mathbf{N}_1} \times \mathbb{R} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \pi_\rho} & & \downarrow \\ P_{\mathbf{N}_1} & \xleftarrow[F']{} & S' & \xrightarrow[\xi]{} & \widetilde{S} & \xrightarrow[\widetilde{F}]{} & \mathbf{N}_1 & \xleftarrow[\varphi]{} & P_{\mathbf{N}_1} \end{array}$$

we see that finding a section of $\widetilde{F}^* \mathbf{E}_\rho$ is the same as finding a function $h : S' \to \mathbb{R}$, such that:

$$\text{(2-4)} \quad h(x_1) = \rho_i h(x_2) \text{ if } F(\xi(x_1)) = F(\xi(x_2)) \in \omega_i \quad \forall\, x_1, x_2 \text{ s.t. } \xi(x_1) = \xi(x_2)$$

and satisfying the analogous conditions on the differentials. We shall denote by σ_h the section of $\widetilde{F}^* \mathbf{E}_\rho$ defined by h. Clearly the following holds:

PROPOSITION 2.25. *The mapping $\pi^* \widetilde{F} \circ \sigma_h : \widetilde{S} \to \mathbf{E}_\rho$ is an embedding, if and only if $(F', h) : S' \to P_{\mathbf{N}_1} \times \mathbb{R}$ is.*

We shall call such a function h a *strange* height for the mapping F.

We introduce some further notations. Put $\widetilde{H}(F) = \big(H(F) \cap \widetilde{S}\big) \cup \partial \widetilde{S}$ and let $H'(F) \subset S'$ be defined by $H'(F) = \xi^{-1}\big(\widetilde{H}(F)\big)$. Denote by $\mathcal{C}$ the set of connected components of $S - H(F)$, by $\widetilde{\mathcal{C}}$ the set of components of $\widetilde{S} - \widetilde{H}(F)$ and finally by $\mathcal{C}'$ the set of components of $S' - H'(F)$. It is immediate that inclusion $\widetilde{S} \hookrightarrow S$ gives a bijection between $\widetilde{\mathcal{C}}$ and $\mathcal{C}$, while $\xi : S' \to \widetilde{S}$ gives a bijection between $\mathcal{C}'$ and $\widetilde{\mathcal{C}}$. For all $C \in \mathcal{C}$ denote by $\widetilde{C} \in \widetilde{\mathcal{C}}$ and by $C' \in \mathcal{C}'$ the corresponding components (i.e. $\widetilde{C}$ is the interior of $C \cap \widetilde{S}$ and C' is the component such that $\xi(C') = \widetilde{C}$).

PROPOSITION 2.26. *Let $C_1,\ C_2 \in \mathcal{C}$. $F(C_1) \cap F(C_2) \neq \varnothing$ if and only if $F(\widetilde{C}_1) \cap F(\widetilde{C}_2) \neq \varnothing$ if and only if $F(C_1') \cap F(C_2') \neq \varnothing$.*

PROOF. The second equivalence is immediate. The first follows from the fact that if $F(C_1) \cap F(C_2) \neq \varnothing$ and $F(\widetilde{C}_1) \cap F(\widetilde{C}_2) = \varnothing$, then $F(C_1) \cap F(C_2) \subset U_\infty$ and the fact that $\partial\big(F(C_1) \cap F(C_2)\big)$ contains some branches of Γ, while $U_\infty \cap \Gamma = \varnothing$. $\square$

PROPOSITION 2.27. *Let $C_1, C_2 \in \mathcal{C}$, then $F(\widetilde{C}_1) \cap F(\widetilde{C}_2)$ and $F'({C'}_1) \cap F'({C'}_2)$ are connected.*

PROOF. Exactly the same as Proposition 2.8. $\square$

Let h be a *strange* height for the excellent map F. The same arguments used in the proof of Proposition 2.13 prove the following:

PROPOSITION 2.28. *Let ${C'}_1,\ {C'}_2 \in \mathcal{C}'$ be such that $F'({C'}_1) \cap F'({C'}_2) \neq \varnothing$. One of the following two holds:*

(1) $\forall x \in {C'}_1,\ y \in {C'}_2$ *if* $F(x) = F(y)$ *then* $h(x) < h(y)$;
(2) $\forall x \in {C'}_1,\ y \in {C'}_2$ *if* $F(x) = F(y)$ *then* $h(x) > h(y)$.

As in the case of the trivial bundle we give a structure of oriented graph to the set $\mathcal{C}$, saying $C_1 \to C_2$ if and only if the corresponding components ${C'}_1,\ {C'}_2 \in \mathcal{C}'$ satisfy condition (1). We call such a graph the graph of h and denote it by $L(h)$.

Remark 2.29. By Proposition 2.26 we see that two components C_1, $C_2 \in \mathcal{C}$ are joined by an edge if and only if $F(C_1) \cap F(C_2) \neq \varnothing$, hence the situation is in all similar to the one we had dealing with the trivial bundle.

As in the case of the trivial bundle, the following is proved:

Proposition 2.30. *The graph $L(h)$ has no loop.*

Put $\Omega^- = \{\omega \in \Omega \mid \rho(\omega) = -1\}$, an analogous of Proposition 2.15 holds in this case too.

Proposition 2.31. *Let h be a strange height for F. If C_1, C_2, C_3, $C_4 \in \mathcal{C}$ and ℓ, ℓ_1, ℓ_2 are proper edges of $H(F)$ the following two hold:*

(A1) *suppose that C_1, C_2 paste along ℓ (i.e. $\bar{C}_1 \cap \bar{C}_2 \supseteq \ell$) and $F(C_3) \cap F(\ell) \neq \varnothing$. Then either $C_1 \to C_3$ and $C_2 \to C_3$ or $C_3 \to C_1$ and $C_3 \to C_2$;*

(B1) *suppose that C_1, C_2 paste along ℓ_1, C_3, C_4 paste along ℓ_2 and $F(\ell_1) \cap F(\ell_2) \neq \varnothing$. Let $r \in R$ be the segment containing $F(\ell_i)$, and suppose that $F(C_1)$, $F(C_3)$ stand on the left side of r while $F(C_2)$, $F(C_4)$ stand on its right side. If $r \notin \Omega^-$, then either $C_1 \to C_3$ and $C_2 \to C_4$ or $C_3 \to C_1$ and $C_4 \to C_2$. If $r \in \Omega^-$, then either $C_1 \to C_3$ and $C_4 \to C_2$ or $C_3 \to C_1$ and $C_2 \to C_4$.*

Proof. Observe that a situation as in (A1) occurs only if $F(\ell)$ is contained either in Γ or in $R - \Omega$, hence the same argument used to prove (A) of Proposition 2.15 proves (A1) too. We now prove (B1). If the segment r containing $F(\ell_1)$ and $F(\ell_2)$ is in $R - \Omega$, the proof is again the same as the one of Proposition 2.15. Suppose $F(\ell_1), F(\ell_2) \subset \omega \in \Omega$ and let $x_1 \in \ell_1$, $x_2 \in \ell_2$ be such that $F(x_1) = F(x_2)$. It is easy to see that we can suppose $x_1, x_2 \in \widetilde{S}$. Let ℓ_1', ℓ_1'' be the edges of $H'(F)$ such that $\xi(\ell_1') = \xi(\ell_1'') = \ell_1$ and let ℓ_2', ℓ_2'' be such that $\xi(\ell_2') = \xi(\ell_2'') = \ell_2$. The assumption in (B1) ensures that C'_1, C'_2, C'_3 and C'_4 verify the following:

$$\begin{gathered}
\partial C'_1 \supseteq \ell_1', \quad \partial C'_2 \supseteq \ell_1'', \quad \partial C'_3 \supseteq \ell_2', \quad \partial C'_4 \supseteq \ell_2'', \\
F'(C'_1) \cap F'(C'_3) \neq \varnothing, \quad F'(C'_2) \cap F'(C'_4) \neq \varnothing.
\end{gathered}$$

Let $x_1' \in \ell_1'$, $x_1'' \in \ell_1''$, $x_2' \in \ell_2'$, $x_2'' \in \ell_2''$ be such that $\xi(x_i') = \xi(x_i'') = x_i$. It is not hard to find four path (see Figure 17) $\gamma_i' : [0,1] \to S'$, $\gamma_i'' : [1,2] \to S'$, such that:

(1) $\quad \gamma_i'(1) = x_i'$ and $\gamma_i''(1) = x_i'' \quad i = 1,2;$

(2) $\quad \forall t \neq 1 \quad \begin{cases} \gamma_1'(t) \in C'_1, & \gamma_1''(t) \in C'_2, \\ \gamma_2'(t) \in C'_3, & \gamma_2''(t) \in C'_4; \end{cases}$

(3) $\quad \forall t \quad \begin{cases} F'(\gamma_1'(t)) = F'(\gamma_2'(t)), \\ F'(\gamma_1''(t)) = F'(\gamma_2''(t)). \end{cases}$

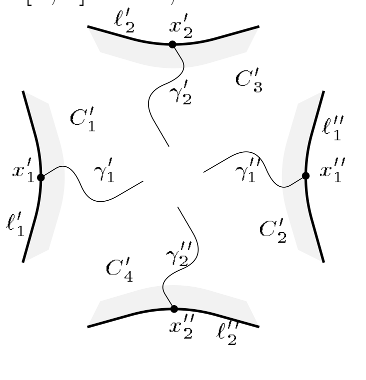

Figure 17

Define:

$$g_i(t) = \begin{cases} h(\gamma_i'(t)) & \text{if } t \in [0,1] \\ \rho(\omega) h(\gamma_i''(t)) & \text{if } t \in [1,2] \end{cases} \qquad i = 1,2.$$

By definition of a *strange* height and (1), we see that g_1 and g_2 are continuous functions. Consider the two cases $\omega \in \Omega^+ = \Omega - \Omega^-$ and $\omega \in \Omega^-$. In the first case, suppose by contradiction that $C'_1 \to C'_3$ and $C'_4 \to C'_3$. By definition of $\to$ and (2), (3), $g_1(t) < g_2(t)$ for all $t < 1$ and $g_1(t) > g_2(t)$ for all $t > 1$. This implies $g_1(1) = g_2(1)$, hence $F'(x'_1) = F'(x'_2)$ and $h(x'_1) = h(x'_2)$ contradicting the fact (F', h) is injective. A similar argument ends the proof in the second case too. □

As in the case of the trivial bundle, Proposition 2.30 and Proposition 2.31 gives necessary and sufficient conditions for the existence of a *strange* height for an excellent map F.

THEOREM 2.32. *Let $F : S \to \mathbf{N}$ be an excellent map, R a system of segments for $F|_\Sigma$ and $\mathbf{E} = \widehat{\mathbf{E}}_\rho$ a line bundle over $\mathbf{N}$. F factors through π if and only if $\mathcal{C}$ can be given a structure of oriented graph with no loops verifying* (A1) *and* (B1) *of Proposition 2.31.*

PROOF. The proof is completely similar to that of Theorem 2.16. □

REMARK 2.33. Repeating almost word by word the statements and proofs in this section, a factorization theorem for generic immersion can be proved. More precisely, let S be a compact surface with boundary, $F : S \to \mathbf{N}$ a generic immersion and R a system of segments for $F|_{\partial S}$. Denote $G(F) \subset S$ the graph made up by ∂S and those arcs in $F^{-1}(R)$ having both vertices in $\partial S \cup F^{-1}(\infty)$. $G(F)$ is the 1-skeleton of a cell decomposition of S such that F restricted to the interior of each cell is one to one. Finally denote by $\mathcal{C}$ the set of 2-cells of such a decomposition.

THEOREM 2.34. *A generic immersion F factors through $\mathbf{E} = \widehat{\mathbf{E}}_\rho$ if and only if $\mathcal{C}$ can be given a structure of oriented graph with no loops verifying* (A1) *and* (B1) *of Proposition 2.31.*

Acknowledgment. This work was partially supported by M.U.R.S.T.

REFERENCES

[1] S. J. BLANK, *Extending immersions of the circle*, Dissertation, Brandeis University, Waltam, Mass., 1967.

[2] V. L. CARRARA ZANETIC, *Extensions of immersions in dimension two*, Trabalhos do Departamento de Matemática **92** (1986), Universidade de São Paulo, Instituto de Matemática e Estatitistica, São Paulo..

[3] C. CURLEY, D. WOLITZER, *Branched immersions of surfaces*, Michigan Math. Jour. **33** (1986), 131–144.

[4] C. L. EZELL, *Branched extensions of curves in compact manifold*, Trans. of the A. M. S. **259** (1980), 533–546.

[5] C. L. EZELL, M. L. MARX, *Branched extensions of curves in oriented surfaces*, Trans. of the A. M. S. **259** (1980), 515–532.

[6] G. K. FRANCIS, *Assembling compact riemann surfaces with given boundary curves and branch points on the sphere*, Ill. Jour. of Math. **20** (1976), 198–217.

[7] G. K. FRANCIS, *Polymersions with nontrivial target*, Ill. Jour. of Math. **22** (1978), 161–170.

[8] G. K. FRANCIS, S. F. TROYER, *Excellent Maps with Given Folds and Cusps*, Houston Jour. of Math. **3** (1977), 165–194.

[9] G. K. Francis, S. F. Troyer, *Continuation: Excellent Maps with Given Folds and Cusps*, Houston Jour. of Math. **8** (1982), 53–59.

[10] R. Z. Goldstein, E. C. Turner, *Extending immersions of the circle*, Preprint, State University of New York at Albany, Department of Mathematics.

[11] A. Haefliger, *Quelques remarques sur les applications differentiables d'une surface dans le plan*, Ann. Inst. Fourier **10** (1960), 47–60.

[12] D. Luminati, *Factorizations of generic mappings between surfaces*, Proc. Amer. Math. Soc. **118** (1993), 247-253.

[13] D. Luminati, *Immersioni di superficie e contorni apparenti*, Dissertazione, Dottorato di Ricerca, Università di Pisa, 1992.

[14] M. L. Marx, *Extensions of normal immersions of* $\mathbf{S}^1$ *into* $\mathbb{R}^2$, Trans. Amer. Math. Soc. **187** (1974), 309–326.

[15] K. C. Millet, *Generic smooth maps of surfaces*, Top. and its Appl. **18** (1984), 197–215.

[16] R. Pignoni, *Curves and Surfaces in Real Projective Space: an Approach to Generic Projections*, Singularities, Banach Center Publication **20** (1988), 335–351.

[17] R. Pignoni, *Projections of Surfaces with a Connected Fold Curve*, Preprint, Università di Milano (1990).

[18] R. Pignoni, *On surfaces and they contours*, Manuscripta Math. **72** (1991), 223–249.

[19] V. Poenaru, *Extension des immersions en codimension 1 (d'aprés Samuel Blank)*, Exposition 342, Séminaire Bourbaki **20** (1967/68).

[20] J. R. Quine, *A global theorem for singularities of maps between oriented 2–manifolds*, Trans. Amer. Math. Soc. **236** (1978), 307–314.

[21] R. Thom, *Les Singularités des Application Diffèrentiables*, Ann. Inst. Fourier **6** (1955–1956), 43–87.

[22] H. Whitney, *On singularities of mappings of Euclidean space.I*, Ann. of Math. **62** (1955), 374–410.

Dipartimento di Matematica, Università di Trento, via Sommarive n. 14, 38050 Povo, Trento, Italy.

E-mail address: luminati@volterra.science.unitn.it, luminati@itnvax.science.unitn.it,